Defence Science Journal, Vol 43, No 1, January 1993, pp 79-86

Modelling the Heterogeneous Markov Attrition Process

Jau-Yeu Menq

Dept of General Course, Chung-Cheng Institute of Technology, Taoyuan, Taiwan

and

Sheng-Lin Chang

Dept of Industrial Management, National Taiwan Institute of Technology, Taipei, Taiwan

ABSTRACT

A model for a heterogeneous dynamic combat as a continuous-time Markov process has been studied, and on account of the special form of its infinitesimal generator, recursive algorithms are derived to compute the important characteristics of the combat, such as the combat time distribution, expected value and variance, and the probability of winning and expected survivors. Numerical results are also presented. This approach can also be used to consider initial contact forces of both sides as random variables.

1. INTRODUCTION

Today the force-on-force attrition models are widely used to evaluate tactics or to appraise equipment. Most of the models are examples of deterministic Lanchester-type equations. Although stochestic models can provide better insight, understanding and modelling flexibility, little attention has been paid because of the computational difficulties.

There are essentially two main approaches to model the force-on-force attritions. One is the Lanchester-type differential equation system. Different combat models are defined by introducing various operational factors. Taylor[1] reviewed the recent developments of the Lanchester-type models of warfare. Since actual combat consists of many different weapons system types, a natural extension is to consider the heterogeneous case. Maybee[2] applied the theory of positive operators to the combined arms models. Taylor[3] discussed an aggregated force model by converting the diverse weapons system types on a side into a single equivalent 'homogeneous' force, then considered the attrition of the two 'derived' forces.

The other method is modelling the force-on-force attrition as a continuous time Markov process. This approach is better for not only representing the combat phenomena, but also deriving the stochastic behaviours. Feigin *et al*[4] developed a simple model describing the dominant features of air combat by a continuous time discrete-space Markov process. Jaiswal[5], and Chang and Menq[6] proposed two different computational methods to derive the results for homogeneous combat forces. Karr[7] discussed the analogical extension to the heterogeneous system and concluded that it is sufficiently intractable.

In this study a computational approach has been proposed to solve the heterogeneous stochastic attrition model. The approach is based on defining a continuous time Markov attrition process and applying the results of matrix-geometric computational algorithms developed by neuts[8]. Since the stochastic model is manipulated in the matrix form and on account of the special form of its infinitesimal generator, some efficient recursive algorithms can be derived, and numerical results are also presented.

Received 5 February 1991, revised 11 February 1992

2. HETEROGENEOUS MARKOV ATTRITION PROCESS

Let us consider a heterogeneous combat between two forces, red and blue. Let M be the number of types of blue combatants, N be the number of types of red combatants, $B_i(t)$ be the number of blue combatants of types i surviving at time t, $B(t) = (B_1(t),\ldots,B_M(t))$, $R_j(t)$ be the number of red combatants of types j surviving at time t, and $R(t) = (R_1(t),\ldots,R_N(t))$.

Furthermore, it is defined that : B_i^U is the maximum possible number of blue combatants of types i, R_j^U is the maximum possible number of red combatants of type j, B_i^E is the surviving blue combatants of type i when blue surrenders, and R_j^E is the red combatants of types j when red surrenders.

For the purpose of simplicity, let $B_i^E = R_j^E = 0$, for all the values of i and j. The states of the Markov process are denoted by :

$$\begin{aligned} &(i_1, \quad , i_M ; j_1, \ldots, j_N), \\ &0 \le i_1 \le B_1^U, \ldots, 0 \le i_M \le B_M^U, \\ &0 \le j_1 \le R_1^U, \ldots, 0 \le j_N \le R_N^U, \end{aligned} \tag{1}$$

Let the state space of the Markov process be E which can be regarded as a model of attrition in combat between a heterogeneous blue side and a heterogeneous red side, provided that the paths $t \to B_i(t)$ and $t \to R_j(t)$ are non-increasing.

Arrange the state space E in the lexicographic order, that is,

$$\begin{aligned} &(B_1^U, \quad .., B_M^U ; R_1^U, \ldots, R_N^U), \\ &(B_1^U, .. \; .., B_M^U ; R_1^U, \ldots, R_{N-1}^U ; R_N^U - 1), .. \\ &B_1^U, \quad .., B_M^U ; R_1^U, \ldots, R_{N-1}^U, 1), \ldots, \\ &(1, \ldots, 1 ; 1, \ldots, 1) \; (0, \quad ,0 ; 0, \quad ,0) \end{aligned} \tag{2}$$

In this model, it is assumed that if any type of force reaches zero, then that force surrenders. Thus, the $(0,\ldots,0;0,\ldots,0)$ state represents an absorption state which stands for the ending condition of the process. In other words, the absorbing state is a set of

$$\begin{aligned} &\{(0, i_2, \ldots \; , i_M ; j_1 \qquad i_N \\ &i_1, 0, \ldots, j_M ; j_1, \qquad , j_N) \\ &(i_1, \ldots, i_M ; j_1, \qquad , j_{N-1}, 0 \; \} \end{aligned} \tag{3}$$

The set of states

$$\begin{aligned} &\{(i, B_2^U, \ldots, B_M^U ; R_1^U, .. \quad , R_N^U), \\ &(i, 1, \ldots, 1 ; 1, \ldots, 1)\}, \quad 0 \le i \le B_1^U \end{aligned} \tag{4}$$

will be called of the level i. The infinitesimal generator of the Markov attrition process can be constructed as

$$A = \begin{vmatrix} T & T^o \\ 0 & o \end{vmatrix} \tag{5}$$

where T is an

$(\prod_{i=1}^{M} B_i^U . \prod_{j=1}^{N} R_j^U)$ square matrix, T° is an

$(\prod_{i=1}^{M} B_i^U . \prod_{j=1}^{N} R_j^U) \times 1$ column vector, 0 is an

$1 \times (\prod_{i=1}^{M} B_i^U . \prod_{j=1}^{N} R_j^U)$ row vector, and o is a scalar.

By writing the elements of the state space of the process as

$$B, R \;=\; B_1, \quad , B_M ; R_1, \ldots, R_N \tag{6}$$

we have the coefficients of the infinitesimal generator A as follows :

$$\begin{aligned} &T((B,R), (B_1,\ldots,B_i - 1,\ldots .B_M \\ &\qquad ;R_1, \; .,R_N)) = \\ &\sum_{j=1}^{N} k_1(i,j) R_j \quad \text{for } i = \quad ,\ldots, M, \end{aligned} \tag{7}$$

$$\begin{aligned} &T((B,R), (B_1 \qquad ,B_M ; R_1,\ldots, R \\ &\qquad\qquad 1, \ldots, R_N \;) = \\ &\sum_{i=1}^{M} k_2(j,i) B_i \quad \text{for } j = \; ,\ldots, N, \end{aligned} \tag{8}$$

$$T^0(B,R) = \sum_{j=1}^{N}\sum_{i=1}^{M} k_2(j,i)B_i + \sum_{i=1}^{M}\sum_{j=1}^{N} k_1(i,j)R_j$$

$$\text{j,s.t. } R_j = 1 \qquad \text{i,s.t. } B_i = 1 \tag{9}$$

$$T((B,R),(B,R)) = -\left(\sum_{i=1}^{M}\sum_{j=1}^{N} k_1(i,j)R_j + \sum_{j=1}^{N}\sum_{i=1}^{M} k_2(j,i)B_i\right) \tag{10}$$

$$T((B,R\ \ (i,j)) = 0, \text{ all other states } (i,j), \tag{11}$$

Where k_1 and k_2 are matrices (of dimensions $M \times N$ and $N \times M$ respectively), representing the killing rates of the combatants of different types of weapons for each of the confronting forces.

Thus, we have $Te+T^o=0$, where e is a column vector with all elements equal to one. The initial probability vector is given by $(a, a_{(0,\dots,0;0,\dots,0)})$, and $ae+a_{(0,\dots,0;0,\dots,0)}=1$. This (aT) representation satisfies the definition of phase-type probability distribution[9].

2.1 Observation

The T matrix can be partitioned as a bidiagonal matrix recursively. This observation is explained briefly here. If the T matrix is partitioned by each level, then we have

$$T = \begin{vmatrix} C & D & & & \\ & C_2 & D_2 & & \\ & & \ddots & & \\ & & & C_{B^U-1} & D_{B^U-1} \\ & & & & C_{B_1^U} \end{vmatrix} \tag{12}$$

where

C_i, $i=1, \ldots, B_1^U$ is an $(\prod_{i=2}^{M} B_i^U \prod_{j=1}^{N} R_j^U$ square matrix, and

$D_i, i=1,\ldots,B_1^U-1($ is an $)(\prod_{i=2}^{M} B_i^U \cdot \prod_{j=1}^{N} R_j^U)$ diagonal square matrix. For each C_i matrices, we can again partition it according to the second type of blue force. For example,

$$C_i = \begin{vmatrix} G_1 & H_1 & & & \\ & G_2 & H_2 & & \\ & & & & \\ & & & G_{B_2^U-1} & H_{B_2^U-1} \\ & & & & G_{B_2^U} \end{vmatrix}$$

where

G_i, $i=1,\ldots,B_2^U$ is an $)(\prod_{i=3}^{M} B_i^U \prod_{j=1}^{N} R_j^U$ square matrix, and

$H_i, i=1,\ldots,B_2^U-1$ is an $(\prod_{i=3}^{M} B_i^U \cdot \prod_{j=1}^{N} R_j^U)$ diagonal square matrix. This partition can be continued until the last type of red force is reached. Thus, a recurrence relationship is developed and will be used in the next section.

To illustrate the current results, we assume that $M=2$, $N=2$ and $B_1^U=3$, $B_2^U=2$, $R_1^U=3$, and $R_2^U=2$. The combat can be described by the square law with the attrition rate matrix as

$$K = \begin{array}{c|cccc|} & \multicolumn{2}{c}{M} & \multicolumn{2}{c}{N} \\ & 1 & 2 & 1 & 2 \\ M_1 & & & 0.3 & 0.2 \\ M_2 & & & 0.5 & 0.5 \\ N_1 & 0.3 & 0.2 & & \\ N_2 & 0.5 & 0.4 & & \end{array}$$

The infinitesimal matrix of this Markov attrition process is presented in Fig. 1.

	(3.2:3.2)	(3.2:3.1)	(3.2:2.2)	(3.2:2.1)	(3.2:1.2)	(3.2:1.1)	(3.1:3.2)	(3.1:3.1)	(3.1:2.2)	(3.1:2.1)	(3.1:1.2)	(3.1:1.1)	(2.2:3.2)	(2.2:3.1)	(2.2:2.2)	(2.2:2.1)	(2.2:1.2)	(2.2:1.1)	(2.1:3.2)	(2.1:3.1)	(2.1:2.2)	(2.1:2.1)	(2.1:1.2)	(2.1:1.1)	(1.2:3.2)	(1.2:3.1)	(1.2:2.2)	(1.2:2.1)	(1.2:1.2)	(1.2:1.1)	(1.1:3.2)	(1.1:3.1)	(1.1:2.2)	(1.1:2.1)	(1.1:1.2)	(1.1:1.1)	(0.0:0.0)
(3.2:3.2)	-6.8	1.6	1.9				1.4						1.9																								0.0
(3.2:3.1)		-5.9		1.9				1.0						1.4																							1.6
(3.2:2.2)			-6.3	1.6	1.9				1.2						1.6																						0.0
(3.2:2.1)				-5.4		1.9				0.8						1.1																					1.6
(3.2:1.2)					-5.8	1.6					1.0						1.3																				1.9
(3.2:1.1)						-4.9						0.6						0.8																			3.5
(3.1:3.2)							-5.8	1.1	1.4										1.9																		1.4
(3.1:3.1)								-4.9		1.4										1.4																	2.1
(3.1:2.2)									-5.3	1.1	1.[illegible]										1.6																1.2
(3.1:2.1)										-4.4		1.4										1.1															1.9
(3.1:1.2)											-4.8	1.1											1.3														2.4
(3.1:1.1)												-3.9												0.8													3.1
(2.2:3.2)													-6.3	1.4	1.6										1.9												0.0
(2.2:3.1)														-5.4		1.6			1.4							1.4											1.4
(2.2:2.2)															-5.8	1.4	1.6			1.0							1.6										0.0
(2.2:2.1)																-4.9		1.6			1.2							1.1									1.4
(2.2:1.2)																	-5.3	1.4				0.8							1.3								1.6
(2.2:1.1)																		-4.4					1.0							0.8							3.0
(2.1:3.2)																			-5.3	0.9	1.1			0.6							1.9						1.4
(2.1:3.1)																				-4.4		1.1										1.4					1.9
(2.1:2.2)																					-4.8	0.9	1.1										1.6				1.2
(2.1:2.1)																						-3.9		1.1										1.1			1.7
(2.1:1.2)																							-4.3	0.9											1.3		2.1
(2.1:1.1)																								-3.4												0.8	2.6
(1.2:3.2)																									-5.8	1.2	1.3				1.4						1.9
(1.2:3.1)																										-4.9		1.3				1.0					2.6
(1.2:2.2)																											-5.3	1.2	1.3				1.2				1.6
(1.2:2.1)																												-4.4		1.3				0.8			2.3
(1.2:1.2)																													-4.8	1.2					1.0		2.6
(1.2:1.1)																														-3.9						0.6	3.3
(1.1:3.2)																															-4.8	0.7	0.8				3.3
(1.1:3.1)																																-3.9		0.8			3.1
(1.1:2.2)																																	-4.3	0.7	0.8		2.8
(1.1:2.1)																																		-3.4		0.8	2.6
(1.1:1.2)																																			-3.8	0.7	3.1
(1.1:1.1)																																				-2.9	2.9
(0.0:0.0)																																					

Figure 1. Elements of the infinitesimal generator.

3. ITERATIVE INVERSION

According to Neuts[9], the expected value and variance of the time until absorption can be computed as

$$\mu_1 = E(X) = -\alpha T^{-1}\mathrm{e}, \mu_2 \quad E(X^2 = 2!\,\alpha T^{-2}\mathrm{e}, \tag{15}$$

and $$Var(X) = E(X^2) - E(X)^2 \tag{16}$$

Since T is very large dimension $\left(\prod_{i=1}^{M} B_i^U \cdot \prod_{j=1}^{N} R_j^U\right)$ square matrix, it will be time-consuming to find its inverse. However, with the observation of the recurrence relationship, one can derive the inverse matrix recursively. Consider a matrix

$$S_1 = \begin{matrix} P_1 & R_1 \\ 0 & P_2 \end{matrix} \tag{17}$$

where P_1 and P_2 are non-singular square matrices of arbitrary dimensions, and R_1 and 0 are matrices of appropriate dimensions. The inversion of the above matrix is

$$S_1^{-1} = \begin{matrix} P_1^{-1} & P_1^{-1} \; R_1 \; P_2^{-1} \\ 0 & P_2^{-1} \end{matrix} \tag{18}$$

Likewise, let

$$S_2 = \begin{vmatrix} P_1 & R_1 & 0 \\ 0 & P_2 & R_2 \\ 0 & 0 & P_3 \end{vmatrix} \tag{19}$$

Inversion of S_2 gives

$$S_2^{-1} = \begin{vmatrix} P_1^{-1} & -P_1^{-1} \; R_1 \; P_2^{-1} & P_1^{-1} \; R_1 \; P_2^{-1} \; R_2 \; P_3^{-1} \\ 0 & P_2^{-1} & -P_2^{-1} R_2 \; P_3^{-1} \\ 0 & 0 & P_3^{-1} \end{vmatrix} \tag{20}$$

$$\begin{matrix} S_1^{-1} & S_1^{-1} \begin{bmatrix} 0 \\ -R_2 \end{bmatrix} P_3^{-1} \\ 0 & P_3^{-1} \end{matrix} \tag{21}$$

By assuming

$$S_k = \left| \begin{array}{cccccc} P_1 & R_1 & & & & \\ & P_2 & R_2 & & & \\ & & & \ddots & & \\ & & & P_k & R_k & \\ & & & & P_{k+1} & \end{array} \right|$$

$$\left| \begin{array}{ccc} & S_{k-1} & \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ R_k \end{bmatrix} \\ 0. & \ldots 0 & P_{k+1} \end{array} \right| \tag{22}$$

By induction, we have

$$S_k^{-1} = \left| \begin{array}{cc} S_{k-1}^{-1} & S_{k-1}^{-1} \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ -R_k \end{bmatrix} P_{k+1}^{-1} \\ 0 & P_{k+1}^{-1} \end{array} \right| \tag{23}$$

To test the computational efficiency of the inversion algorithm, it is compared with the direct inversion by using the 386-MATLAB which is known for the high performance interactive ability. Each P_k s and R_ks are generated as a 20 × 20 random number matrices. The elapsed computation time is calculated by the MATLAB built-in function, *e* time, and the numbers of addition, multiplication or division, flop. Table 1 shows the test results, in which the three different numbers in the rows of iterative algorithm represent, (i) the accumulated numbers from *k*-1, (ii) the inversion of P_k^{-1} and (iii) the multiplication of S_k^{-1} [0,...,0,-R_k] P_{k+1}^{-1}. From the table, it is seen that the saving of computation time and flops from the iterative algorithm is very significant for large size matrices.

4. ANALYSIS OF THE HETEROGENEOUS MARKOV ATTRITION PROCESS

As the heterogeneous Markov attrition model is defined, some useful computational algorithms can be developed. The *a* vector specifies the initial contact forces for both sides. Thus, the initial confronting enemy forces can be viewed as the random variables. Without the loss of generality, let $a_{(0,\ldots,0;0,\ldots,0)}=0$ i.e., the probability that either side surrenders without any engagement in the beginning of combat is zero.

For the heterogeneous Markov attrition model, the R_is matrices are diagonal square ones. If the square law process is assumed, then the R_is are the same. With the derivations of the iterative inversion algorithm and the

Table 1. The comparison of direct inversion and iterative inversion

K		1	2	3	4	5	6	7	8	9	10	11	12
Dimension		20^2	40^2	60^2	80^2	100^2	120^2	140^2					
CPU* Time	Direct	0.11	0.83	2.43	5.60	11.10	17.97	32.19	41.63	69.53	80.47	126.61	160.49
	Iterative	0.00	0.11	0.39	0.88	1.70	2.96	4.71	8.12	11.97	15.92	20.92	26.90
		0.11	0.11	0.11	0.11	0.16	0.16	0.11	0.16	0.11	0.17	0.11	0.17
		0.00	0.17	0.38	0.71	1.10	1.59	3.30	3.69	3.84	4.83	5.87	7.08
		0.11	0.39	0.88	1.70	2.96	4.71	8.12	11.97	15.92	20.92	26.90	34.15
Flops**	Direct	18	135	448	1052	2044	3519	5574	8303	11803	16174	21506	27898
	Iterative	0	18	68	182	392	730	1228	1918	2832	4002	5460	7238
		18	18	18	18	18	18	18	18	18	18	18	
		0	32	96	192	320	480	672	896	1152	1440	1760	2112
		18	68	182	392	730	1228	1918	2832	4002	5460	7238	9368

*In seconds, **in thousands

special structure of T matrix, T^{-1} can be found with the reasonable time. Another computational efficiency can be achieved by the relevant elements of a vector. For instance, if $a=(1,0,\ldots,0)$, i.e., both initial forces are known, then only the elements of the first row can be computed for T^{-1} to find the expected absorption time. According to Neuts[9], the distribution of the time until absorption in state $(0,\ldots,0)$ given the initial probability vector $(a\ a_{(0,0\ldots,0;0,\ldots,0)})$ is

$$F(x) = 1 - \alpha \exp (Tx)e, \ x \geq 0 \qquad (24)$$

A built-in function of exp-m (Tx) in MATLAB can be used to derive the $F(x)$.

From the infinitesimal generator, we can compute the transition probability of the imbedded Markov chain. An acyclic network can be drawn to represent the attrition process. The source states correspond to the states of non-zero elements of the a vector, and the sink states are those with any one of the B_js or R_js equal to zero, let $\Pi(B,R)$ be the transition probability of the sink states, then the probability of blue wins is the summation of $\Pi(B,R)$ for all possible (B,R) where one of the R_js equal to zero, $j=1,\ldots,N$. The probability of red wins is the summation of $\Pi(B,R)$ for all possible (B,R) where one of the B_is equal to zero, $i=1,\ldots,M$.

The expected survivors stand for the available force for the next battle or can be used to appraise the expected loss. As a field commander, this information provides a way to evaluate the quality or effectiveness of engaging strategies. From the results of $\Pi(B,R)$, we can compute the expected survivors of each type given winning the battle.

5. EXAMPLE

To demonstrate the results, an example of a heterogeneous combat between two forces, blue and red, is taken to show the computations. Each force consists of two weapons companies (WC), two combat engineering companies (EC) and four infantry companies (IC). The red field commander decides to deploy all troops in attack formation. As for the blue field commander, the following three strategies are considered :

S1: *Attack strategy* : deploying all troops in attack formation, intend to destroy the enemy as soon as possible.

S2: *Surround strategy* :deploying half force in defence position and the other circling behind the enemyline, intend to trap the enemy while levitating own loss, and

S3: *Stall strategy* : deploying all force in defense position, to get the enemy stuck as long as possible.

For the three difference strategies, the appropriate attrition rates are determined from Table 2. The sizes of the state space for the three strategies are 257, 33 and 257. Figure 2 shows the combat time distributions of each stretegy. Table 3 gives the expected combat time, variance, the probability of winning for each force and the expected survivors given winning. Figure 3 shows the probabilities of victory against survivors of each force for different strategies. From the above information, the decision-making process can be incorporated with the methods of multiple objectives optimisation.

6. CONCLUSION

The heterogeneous continuous time Markov process is considered sufficiently intractable by many authors.

Table 2. The attrition rates

Linear law (for defending situation)							Square law (for attacking situation)						
	RWC	REC	RIC	BWC	BEC	BIC		RWC	REC	RIC	BWC	BEC	BIC
RIC				0.01	0.08	0.02	RIC				0.15	0.30	0.25
REC				0.01	0.05	0.01	REC				0.10	0.25	0.15
RWC				0.05	0.10	0.02	RWC				0.20	0.40	0.35
BIC	0.01	0.01	O.01				BIC	0.18	0.30	0.25			
BEC	0.01	O.04	0.01				BEC	0.12	0.25	0.32			
BWC	0.05	0.10	0.02				BWC	0.25	0.50	0.40			

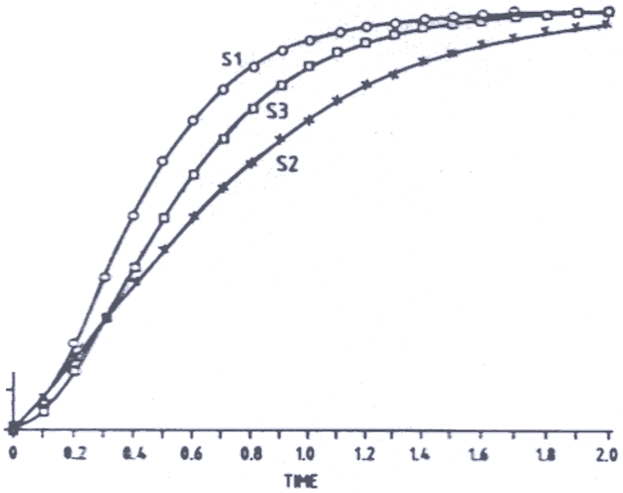

Figure 2. Combat time distribution of each strategy

Thus, either a simplified method or a simulation approach is recommended to derive the approximate results. In this study, we have proposed an approach to compute the direct solutions. The approach is based upon modelling the attrition phenomena as a finit Markov process which satisfies the definition of phase-type distribution. By observing the special structure of the infinitsimal generator, recursive algorithms were deserved such that the intended stochastic behaviour can be obtained.

An important feature of this approach is that the initial contact force level need not be known when the combat begins. The a vector specifies that initial probability of the involving forces. Thus, the randomness of the encountering enemy force can be

Table 3. The computation

	Expected combat time	Variance	Probability of winning		Expected survivors given winning					
			Blue	Red	BWC	BEC	BIC	RWC	REC	RIC
S1	0.4421	0.0882	0.5562	0.4438	1.75	[illegible]	3.39	[illegible]	[illegible]	3.30
S2	0.7158	0.3287	0.6730	0.3270	1.00	[illegible]	1.94	[illegible]	.56	2.91
S3	0.5634	0.1404	0.9823	0.0177	1.96	[illegible]	3.96	[illegible]	[illegible]	2.36

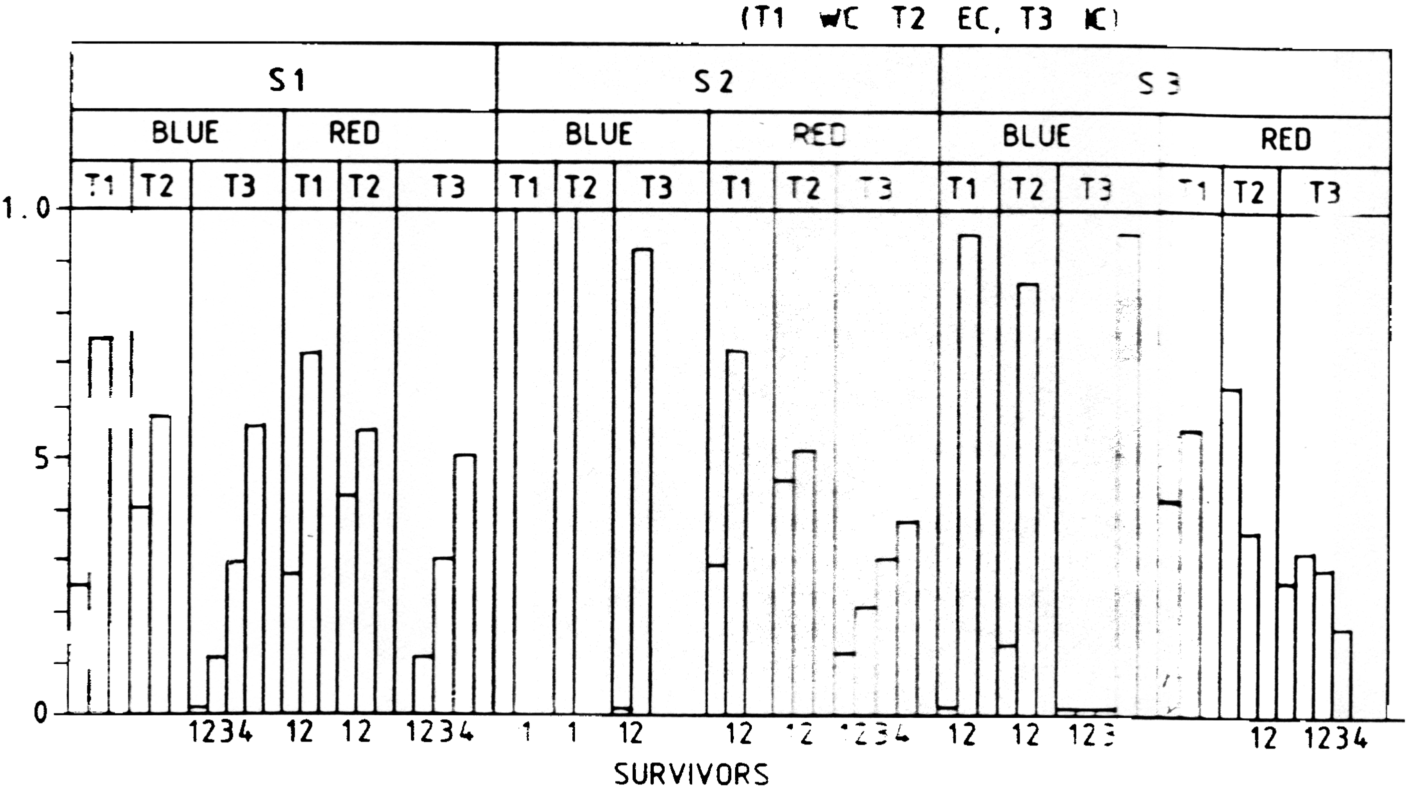

Figure 3. Probabilities of victory against survivors

analysed. the allocation of fires can be easily achieved by changing the elements of T matrix accordingly.

Future research can be extended to the optimal strategy of using tactical reserves, deploying policy of the forces, or transferring combat units. Futhermore, this stochastic attrition model can be accomodated to the existing war game theater-level combat models to develop them as a decision support system for field commanders.

REFERENCES

1. Taylor, J.G. Force-on-force attrition modelling; military applications-section. Operations Research Society of America, Baltimore, 1981.
2. Maybee, J.S. The theory of combined-arms Lanchester-type models of warfare. *Naval Reaearch Logistics Quarterly*, 1985, **32**, 225-37.
3. Taylor, J.G. A Lanchester-type aggregated-force model of conventional ground combat. *Naval Research Logistics Quarterly,* 1983, **30**, 237-60.
4. Feigin, P.D.; Pinkas, O & Shinar, J. A simple model of the analysis of multiple air combat. *Naval Research Logistics Quarterly,* 1984, **31**, 413-29.
5. Jaiswal, N.K. Probabilistic analysis of combat models. *Annals of Operations Research,* 1987, **9**, 561-73.
6. Chang, S.L. & Menq, J.Y. Modelling a Markov attrition process. *Defence Science Journal,* 1989, **39**(2), 211-20.
7. Karr, A.F. Lanchester attrition processes and theater-level combat models. *In* Mathematics of conflict, edited by Martin Shubik. Elsevier Science Publishers, New York, 1983.
8. Neuts, M.F. The probabilistic significance of the rate matrix in matrix-geometric invariant vectors. *Journal of Applied Probability,* 1980, **17**, 291-96.
9. Neuts, M.F. Matrix-geometric solution in stochastic model: an algorithmic approach. Johns Hopkins series in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, 1981.