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Optimization Problem of a Sea-vehicle Entry into Water

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ABSTRACT

For a vehicle diving into the sea, the variation of velocity with horizontal distance covered was studied, the conditions being that the vehicle requires minimum time and that the horizontal distance is fixed. The variation of the vehicle's trajectory angle and time versus depth was also investigated. The variational problem makes use of the isoperimetric condition. The second-order differential equation was solved by the Runge-Kutta-Nyström method and the isoperimetric condition by Simpson's rule.

1. INTRODUCTION

Several workers have studied the entry of a space vehicle into a planetary atmosphere. Marinescu[1] studied the case of a vehicle entering the Earth's atmosphere, which required minimum time to descend from a higher altitude to a lower one[2], given a specified horizontal distance. Since the methods adopted in the study of submarine problems rely heavily on, and are almost identical to, those used by aerodynamicists, it is proposed to apply the above aspect of re-entry problem to an underwater vehicle as it plunges from one depth to another. In this paper, 'vehicle' essentially refers to a submarine though a submersible or a torpedo could also come under its purview.

When a body moves through a fluid it appears to have a greater mass than its actual mass[2,3]. The effect of added mass has therefore to be considered while analysing motion in an underwater environment. Added masses for regular solids, like cylinder or a sphere, have been dealt with in treatises on classical hydrodynamics[3]. If the sea-vehicle is assumed to be a submarine, which is a right circular cylinder, then the presence of the liquid increases the effective inertia of the cylinder by an amount $M' = \pi a^2 \rho$, where M' is the amount of the liquid displaced by the cylinder (of unit length); a, its radius and, ρ the density of the liquid. Thus, if X is the extraneous force parallel to the axis of x and U, the velocity of the cylinder, then the equation of motion is $d/dt\,((1/2)\,MU^2 + (1/2)\,M'U^2) = XV$. If, on the other hand, the sea-vehicle is assumed to be a bathyscaphe, which is spherical, then the liquid increases the inertia of the sphere by half of the mass of the liquid displaced, i.e. by $M'/2$, where M' equals $(4/3)\,\pi a^3$ and a is the radius of the sphere. Thus, if V is the velocity of the sphere, then the equation of motion can be written as $d/dt\,((1/2)\,MV^2 + (1/4)\,M'V^2) = XV$. Incidentally, in aerodynamic studies, the effect of added mass is usually very small and is therefore neglected.

2. EQUATIONS OF MOTION

The two dimensional equations of motion for a non-thrusting vehicle, entering stationary ocean, when gravity and buoyancy forces along with lift and drag forces are considered, is given by

$$(m + m')\,\dot{V} = -\frac{SC_x \rho V^2}{2} + (m - \rho v)\,g\sin\theta \qquad (1)$$

$$(m + m')\,V\dot{\theta} = -\frac{SC_z \rho V^2}{2} + (m - \rho v)\,g\cos\theta \qquad (2)$$

$$\dot{z} \quad -V\sin\theta \qquad 3)$$

$$\dot{x} = \quad V\cos\theta \qquad (4)$$

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where m is the mass of the diving vehicle; m', added mass; V, velocity of vehicle; S, reference area; C_x, drag coefficient; v, volume of the vehicle; g, acceleration due to gravity (assumed constant); θ, inclination of the trajectory with the horizontal at instant of time t; C_z, lift coefficient and the dot over V, θ, z and x represents differentiation with respect to time.

From Eqns (1) and (3), we get

$$\sin\theta = \frac{g\,V^2}{(VV' + gb)} \tag{5}$$

where $a = \dfrac{SC_x\rho}{2(m+m')}$, $b = \dfrac{m-\rho v}{m+m'}$, and $V' = \dfrac{dV}{dz}$

From Eqns (3) and (4), we get

$$\frac{dx}{dz} = -\cot\theta \tag{6}$$

$$x = \int_{z_f}^{z_i} \frac{\left[1-\left(\dfrac{a\,V^2}{VV'+gb}\right)^2\right]^{1/2}}{\left(\dfrac{g\,V^2}{VV'+gb}\right)}\,dz = l \tag{7}$$

where z_i is the initial depth and z_f the final depth (positive in the upward direction).

For a vehicle which requires minimum time one has to find the minimum of the functional

$$t = \int^{z_i} \frac{\left(VV'+gb\right)}{a\,V^3}\,dz \tag{8}$$

The variational problem now requires that the minimum of the functional (8) be determined given the isoperimetric condition (7).

The curve which achieves the extremum of the functional (8) is an extremum of the auxiliary functional[4].

$$J = \int_{z_f}^{z_i} H\,dz \tag{9}$$

where

$$H = \frac{VV'+gb}{a\,V^3} + \lambda\,\frac{\left[1-\left(\dfrac{a\,V^2}{}\right)^2\right]^{1/2}}{\left(\right)} \tag{10}$$

Making use of Euler's equation

$$\frac{\partial H}{} - \frac{d}{dz}\left(\frac{\partial H}{\partial V'}\right) = 0 \tag{11}$$

one gets the differential equation after simplification

$$V'' = \frac{3\,gb\left[\,(VV'+gb)^2 - (aV^2)^2\,\right]^{3/2}}{\lambda\,a^2V^8}$$
$$+ \frac{(3\,gb\,VV' + V^2V'^2 + 2\,g^2b^2)\left[(VV'+gb)^2-(aV^2)^2\right]}{a^2\,V^7}$$
$$\frac{(3\,gb\,V' + 2VV'^2)\left[(VV'+gb)^2-(aV^2)^2\right]}{a^2\,V^6}$$
$$+ \frac{(3\,gb\,VV' + V^2V'^2 + 2\,g^2b^2)\,(VV'^2 + gbV')}{a^2\,V^6} \tag{12}$$

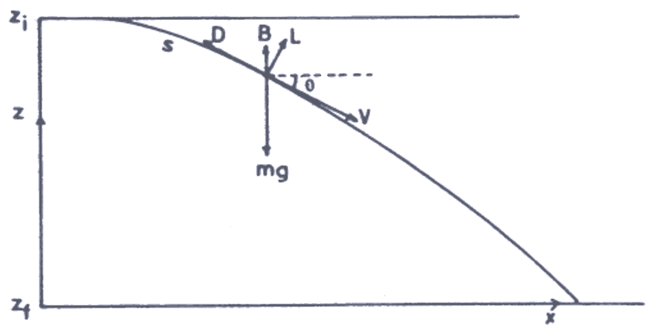

Figure 1. Geometry of the sea-entering vehicle.

3. NUMERICAL SOLUTION

The second-order differential equation was numerically integrated by the method of Runge-Kutta-Nyström. For V_i = 50 m/s, a = 0.002, b = 0.1, z_i = 1000 m and z_f = 0 m the variation of velocity with depth is shown in Fig. 2 for l = 500 m and 600 m. The velocity decreases rapidly with depth initially but later, gradually.

Equation (7) was integrated by Simpson's Rule and the variation of the vehicle's depth with horizontal

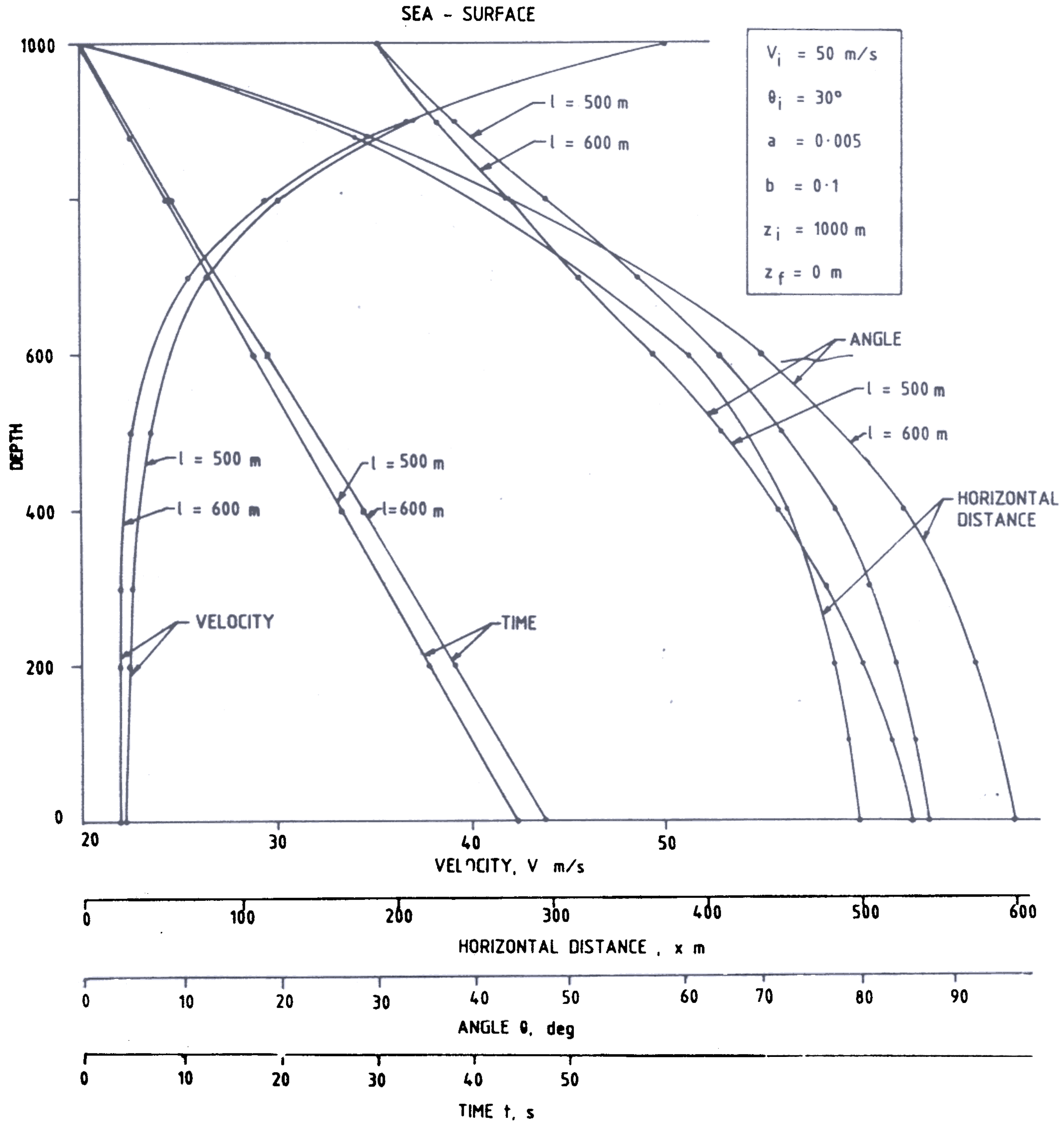

Figure 2. **Variation of velocity, horizontal distance, trajectory angle and time with depth.**

distance travelled is shown in Fig. 2 for l = 500 m and 600 m. Here the tendency for rapid decrease of depth with horizontal distance covered is evident only later. As the vehicle sinks, its velocity decreases. This is essentially due to the choice of the constants which make the drag term on the RHS of Eqn (1) to dominate over the gravity term. If the initial velocity is considered as small, then the gravity term will dominate and the vehicle will sink with increasing speed.

Figure 2 shows the variation of the trajectory angle from an initial horizontal entry angle of 30° to about 87° for l = 500 m. For l = 600 m, for the same horizontal angle, the final angle is about 86°.

Figure 2 also depicts the time required by the vehicle to go from one depth to another. For l = 500 and 600 m the variation is almost linear.

The effect of λ on l is shown in Fig. 3. Since for higher values of λ there is little change in l, it is represented on a logarithmic scale. Figure 3 also shows that varying λ cannot give all conceivable values of l; particularly, values below 400 m are not feasible.

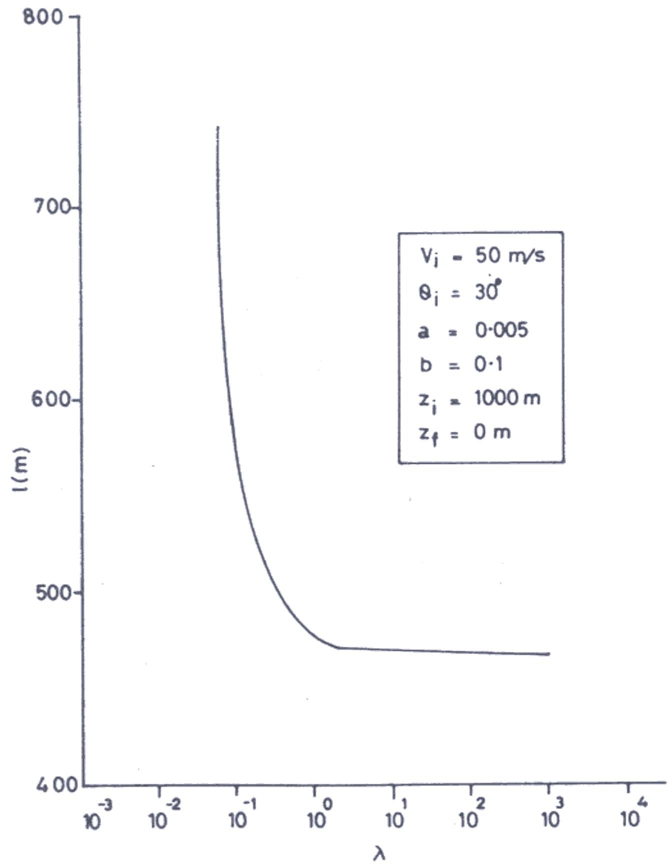

Figure 3. Variation of horizontal distance with λ.

4. DISCUSSION AND CONCLUSIONS

The solution of Eqn (12) provides the minimum of the functional (8) because the Legendre's condition of a weak minimum $H_{V'V'} > 0$ is satisfied. This condition is valid when λ is negative. If λ is positive, then $H_{V'V'} < 0$; thus one gets a maxima instead of a minima. The importance λ of is clear at this stage.

We have chosen the constants arbitrarily. It would be better if realistic data are used. The main hurdle in such a case is that a literature void exists owing to the tendency of researchers to submit their findings to inaccessible technical journals or to even more elusive defence reports. We however, feel that the efficacy of the analysis is not impaired because of the choice of the constants.

The minimum time problem cannot be solved by taking any arbitrary values; rather one has to exercise caution in using the values. For example, in the above problem, the solution for l=100 m (say) cannot be obtained because of the fact that the initial angle θ_i has been considered as 30°. Probably, a very steep angle is required to realise the objective and before making a choice its physical validity needs to be examined. This argument is valid, perhaps to a greater extent, for other parameters.

Besides minimising the time, the distance travelled, s, can also be minimised. In that case the LHS of Eqns (1) to (4) would be $(m + m')\ V\ dV/ds$, $(m + m')\ V^2\ d\theta/ds$, $V\ dz/ds$ and $V\ dx/ds$, respectively. The differential Eqn (12) would necessarily have to be modified. We propose to deal with actual systems and realistic constants in our future study.

The numerical calculations were done on a DCM TANDY 3000 PC at ISSA.

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REFERENCES

1 Marinescu, A. Isoperimetric formulation for some problems of optimisation of the entry into

atmosphere. *Am. Inst. Aero. Astro. J.*, Sept 1973, **11**(12), 1768-70.

2. Spencer, J.B. Stability and control of submarines : Parts I-IV. *J. Naval Sci. Services*, 1968, **23**(3), 187-205.

3. Besant, W. H. & Ramsey, A. S. A Treatise on hydrodynamics: Part II – Hydrodynamics. Neil and Co. Ltd., Edinburgh, 1962. pp. 92-126.

4. Elsgolts, L. Differential equations and the calculus of variations. Mir Publishers, Moscow, 1970. pp. 389-407.

5. Kreyszig, E. Advanced engineering mathematics. Wiley, New York, 1972. pp. 830-69.