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Design & Application of Economical Process Control Charts

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ABSTRACT

Statistical quality control techniques are useful in monitoring the process behaviour. Attribute control charts are widely used in process control. The selection of sample size, sampling interval and control width of the control chart is important in minimising the quality costs. Control chart parameters, like 3σ control limits and fixed fraction sampling at conveniently selected sampling intervals result in deplorable cost penalties in quality control. The best selection of these parameters depends on several process parameters, like frequency of occupancy of a shift in the process, cost of sampling, cost of investigation for finding assignable cause, probability of false alarms, penalty cost of defectives and process correction costs.

A general model has been developed to determine the total quality cost as a function of these parameters. Probability of not identifying a process shift (β-risk) and probability of wrongly concluding that the process got shifted (α-risk) are considered in developing the model. This cost equation is optimised to determine optimum values of control chart parameters. Fibonacci search is used to quicken the analytical method for determining optimum sampling size and control width. The proposals made by Duncan, Montgomery, Gibra and Chiu for determining the optimum control chart parameters are critically examined and compared with the present model. Case studies were conducted in two foundries. Optimum control chart parameters in casting of cylinder liners and cast plates are determined. It has been found that quality costs are considerably reduced by using optimally designed control chart parameters with proposed method.

NOMENCLATURE

a	Fixed cost of sampling
b	Cost per unit sample
g	Expected inspection and charging time per unit sample
h	Sampling frequency
k	Width of control limits in a number of standard deviations
M	Hourly penalty cost of operating in the out-of-control state
n	Sample size
p	Power of the control chart

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t_0	Expected search time for a false alarm
t_1	Expected time to find assignable cause
t_2	Expected time to repair the process
Y	Cost of false alarm
W	Cost to locate and repair the assignable cause
α	Type-I error probability
β	Type-II error probability
δ	Shift in the process for a number of standard deviations
λ	1/Mean in-control period.

1. INTRODUCTION

Shewart invented the control charts in 1924. Thereafter, the control charts are widely used to establish and maintain statistical control of a process. These are also effective devices for estimating process parameters and analysing process capability. The use of a control chart requires selection of a sample size, a sampling frequency, i.e. interval between samples, and the control limits for the chart. The selection of these parameters is called the design of a control chart.

Traditionally, control charts have been designed with respect to statistical criteria only. This usually involves selecting the sample size and the control limits so that the power of the test to detect a particular shift in the quality characteristic and the type-I error probability are equal to specified values. The design of a control chart has economic consequence in that the costs of sampling and testing, costs associated with investigating out-of-control signals and possibly correcting assignable causes, and costs of allowing non-conforming units to reach the consumer, are all affected by the choice of the control chart parameters. Therefore, it is logical to consider the design of control chart from an economic viewpoint.

The economic design of control charts was introduced by Duncan[1]. He proposed an economic model for the optimum design of the $\bar{X}$ chart. Ladany[2] followed the economic design of p-charts. Chiu[3] formulated a cost model of the np-chart, using the variation of Duncan's $\bar{X}$ chart model. Along with developments in control charts, developments and improvements on the economic design have continued. Montgomery[4] listed 51 references on the topic.

This paper presents a general model for determining the economic design of control charts. The model applies to all control charts, regardless of the statistic used. It is necessary to calculate the average run-length of the statistic assuming the process to be in-control and also assuming the process to be out-of-control in some specified manner. The cost function is derived and the total cost of quality is minimised. The assumptions considered while developing the model are discussed. A numerical technique that can be used to minimise the cost function is given. An initial approximation is derived and an iterative search technique is used to achieve optimum design parameters, which leads to minimum process control cost. The case studies were performed in two foundries.

2. FORMULATION OF OBJECTIVE FUNCTION

A production cycle and the time between the start of successive in-control periods are defined. The process is assumed to start in a state of in-control. When the process is disturbed by the assignable cause, it is said to be in the state of out-of-control. Samples of size n are drawn every hours and control chart is drawn. When an alarm is given by the control chart, a search for the assignable cause is undertaken and the process repaired.

2.1 Cycle Time

The cycle time is the sum of the following:

(a) The time until the assignable cause occurs in a process which is in-control normally.

(b) The time until the next sample is taken.

(c) The time to take the sample and interpret the results.

(d) The time until the chart gives out-of-control signal.

(e) The time to locate the assignable cause and repair the process.

The in-control time is assumed as a negative exponential random variable with mean $1/\lambda$. If production continues during the search state, the average time for occurrence of the assignable cause is simply $1/\lambda$. If production ceases during search state, the average time until the assignable cause occurs is $1/\lambda$ plus the time spent during false alarms. Let t_o be the expected search time for a false alarm. Let α be the type-I error probability or probability of a point falling outside the control limits, when the process is in the state of in-control. The expected number of false alarms per production cycle before the process goes out-of-control is just α times the expected number of samples taken in the in-control period.

The expected number of samples taken while the process in-control is s[1].

$$S = \sum_{i=0}^{\alpha} \int_{(i)h}^{(i+1)h} i\,\lambda\, e^{-\lambda x}\, dx$$

$$= \frac{e^{-\lambda h}}{1 - e^{-\lambda h}}$$

Assuming that the assignable cause occurs between i^{th} and $(i+1)^{th}$ samples. Then the expected number of false alarms per cycle

$$B_O = \frac{\alpha \cdot e^{-\lambda h}}{1 - e^{-\lambda h}}$$

The total time of false alarms per cycle is

$$t_o . B_o \tag{1}$$

The expected time until the occurrence of assignable cause, the in-control period is

$$1/\lambda + (1-\delta_1) . t_o . B_o \tag{2}$$

Where, $\delta_1 = 1$, if production continues during search and $\delta_1 = 0$, if production ceases during search.

Let p be the probability of a true alarm when the process is in the state of out-of-control, under the influence of assignable cause. p can also be called as the power of the chart and is equal to $(1-\beta)$, where β is type-2 error probability or probability of falling inside the control limits when the process is in the state of out-of-control. When the assignable cause occurs, it may occur at any time between two samples, say the i^{th} and $(i+1)^{th}$ samples. Then the expected time of occurrence within this is equal to τ[1]

$$\tau = \frac{\int_{i.h}^{(i+1).h} \lambda\, e^{-\lambda x}\, (x - i.h)\, dx}{\int_{i.h}^{(i+1).h} \lambda\, e^{-\lambda x}\, dx}$$

$$= \frac{1 - (1 + \lambda h) . e^{-\lambda h}}{\lambda (1 - e^{-\lambda h})}$$

The expected time before a sample statistic falls outside the control limits is

$$(h/p - \tau) \tag{3}$$

Let g be the expected time to collect a sample and chart per unit. For a sample of n units, the time to take the sample and interpret the results is given by $g.n$.

The expected time from the occurrence of assignable cause to declare the process to be out-of-control state will be

$$h/p - \tau + g.n \tag{4}$$

Let t_1 be the expected time to find the assignable cause and t_2 be the expected time to repair the process.

The expected out-of-control period is

$$B = h/p - \tau + g.n + t_1 + t_2 \tag{5}$$

The expected cycle time, $E(T)$

= In-control time + out-of-control time

$$= 1/\lambda + (1-\delta_1) . t_o . B_o + B \tag{6}$$

2.2 Cost Function

The following categories are customarily considered in the development of an economic model:

(a) The costs of sampling and testing.

(b) The costs associated with the investigation of an out-of-control signal and with the repair or correction of any assignable cause found.

(c) The costs associated with the production of non-conforming items.

The cost function represents the total expected cost per cycle. Let a be the fixed cost per sample and b be the cost per unit sample. Then the expected inspection and sampling cost is given by $(a+b.n)$(production time)/(sampling frequency).

Production time depends on whether or not production continues during search or repair.

Let $\delta_1 = 1$, if production continues during search,

$\delta_1 = 0$, if production ceases during search,

$\delta_2 = 1$, if production continues during repair,

$\delta_2 = 0$, if production ceases during repair.

Production time $= 1/\lambda + D$

where,

D = out-of-control period production time

$= h/p - \tau + g.n + \delta_1 . t_1 + \delta_2 . t_2$

The expected cost of sampling per cycle,

$$C_1 = \frac{(a+b.n)}{h}(1/\lambda + D) \tag{7}$$

Let Y be the cost per false alarm which includes the cost of search and testing plus the cost of down-time if production ceases during search. Let W be the cost for locating and repairing the assignable cause which also includes the cost of down-time, if production ceases during search or repair. Then the expected cost for false alarms, and locating and repairing the assignable cause

C_2 = Number of false alarms × cost per false alarm + cost for locating and repairing the cause

$$= B_o.Y + W \tag{8}$$

where

B_o is the number of false alarms.

Let M be the penalty cost per hour owing to a greater percentage of items being defective due to occurrence of assignable cause.

The expected cost per cycle arising from out-of-control condition

$$C_3 = M \times D \tag{9}$$

The total expected cost per cycle

$$E(C) = C_1 + C_2 + C_3$$

$$= \frac{(a+b.n)}{h}(1/\lambda + D) + (B_o . Y + W) + M.D \tag{10}$$

Dividing the expected cost per cycle by cycle time gives the expected cost per hour C_1 and thus cost function

$$C = \frac{E(C)}{E(T)}$$

$$= \frac{\frac{(a+b.n)}{h}(1/\lambda + D) + B_o . Y + W + M.D}{1/\lambda + (1-\delta_1) . t_o . B_o + B} \tag{11}$$

3. ASSUMPTIONS IN THE MODEL

The first assumption made is that time of in-control period is a negative exponential random variable with mean $1/\lambda$. This implies a memoryless process. On esoteric grounds, this is perfectly reasonable, as the occurrence of assignable cause is a random event. Events such as tool wear exhibit predictability, the average wear should be subtracted to obtain independent events. If a different distribution is assumed, both τ and the average time in-control will change. In general, τ will depend on where the shift occurs. Since, $0 < \tau < h$, however, the change will have a minor effect. On the other hand, the average time required for the assignable cause to occur can change drastically even if the distribution of time in-control has mean $1/\lambda$. One reason is that the process starts anew after each false alarm. Depending upon the distribution, false alarms can increase the time until the assignable cause occurs. A memoryless process is unaffected by false alarms. If we assume that the process continues after a false alarm, as if the false alarm never occurred, then the average time in-control is unchanged by false alarms.

The other major assumption is that there is only one assignable cause with a shift by known amount. Several authors, including Duncan[5], Chiu[6], and Gibra[7], have considered the cases in which there were many assignable causes. These authors concluded that a single assignable cause model with weighted average shift and weighted average time of in-control closely approximated the multiple cause model.

4. MINIMISATION OF OBJECTIVE FUNCTION

The goal of the economic design of control charts is to find the optimum sample size n, sampling frequency h and width of control limits k to minimise the total expected cost per hour as per Eqn. (11). The width of control limits, i.e. k, affects α and β risks. The objective function, expected cost per hour is a function of the design parameters n, h and k. Fibonacci search technique is used to iterate on k and n. Since the cost functions are continuous parameters of h, the partial derivatives of the expected total cost function with respect to h were set equal to zero. Several approximations were tried to obtain a simple and quite accurate expression for optimal sample interval h, for a specified pair (n,k). Chiu[8] proposed approximations for τ and number of false alarms B_0. They are :

$$\tau = h/2$$

$$B_o = \alpha \, . \, [(h.\lambda)^{-1} - 1/2 + h.\lambda/12]$$

In cycle time, when compared to the in-control period, search time for false alarms and out-of-control period are small and can be neglected in cost equation[2].

Rewriting the cost function after neglecting false alarms, search time and out-of-control time

$$C = \frac{\dfrac{(a+b.n)}{h}(1/\lambda + D) + B_o \, . \, Y + W + M.D}{1/\lambda}$$

$$= \lambda.M.D + \lambda.B_o \, .Y + \lambda.W + \frac{(a+b.n)}{h}(1 + \lambda\,.D)$$

Using Chiu's approximations and ignoring all terms containing powers of τ greater than 1, and equating the partial derivative of the total expected cost per hour with respect to h to zero i.e $\partial C/\partial h = 0$, the initial root for h is

$$h = \sqrt{\frac{(\alpha\,.Y + a + b.n)}{\lambda\,.M.(1/p - 1/2)}} \qquad (12)$$

4.1 Algorithm to Find Optimum Design Parameters

The algorithm to find optimum design parameters is given below.

Let (n_1, n_2) be the range for sample size n and (k_1, k_2) be the range for width of control limits k

(both user defined)

Find h_o i.e. initial root (satisfying Eqn (12))

Iterate on h using Newton-Raphson method

Iterate on k using Fibonacci search

Iterate on n using Fibonacci search

A Fibonacci search technique is used to find the optimum sample size n in given range. The initial settings of n_1 and n_2 are used to interpolate n_3 and n_4 based on the Fibonacci search procedure described by Sugie[10]

$$n_3 = n_2 - (n_2 - n_1)/r$$

$$n_4 = n_1 - (n_2 - n_1)/r$$

where

$$r = (1 + \sqrt{5})/2$$

For each sample size n (i.e. for n_3 and n_4) the optimum values of k and h and the total expected cost per hour are found. If cost/hour with n_3 is less than that with n_4, the new interval will be n_1, n_4.

Otherwise, the new interval will be n_3, n_2. The initial settings will be replaced by new interval. The same procedure is applied to the new interval. The search will be terminated when the difference between the revised n values is ≤ 2. The optimum set of n, k, h is finally determined by finding the minimum of the function among these parameters.

For the optimum value of k for a given sample size n, a Fibonacci search procedure is used with user defined initial values. For each pair n, k the differential $\partial c/\partial h$ is equated to zero to solve for new h. Considering all the terms and without neglecting out-of-control period and false alarms, the partial derivation of cost function with respect to h can be written as,

$$\frac{\partial c}{\partial h} = \frac{1}{DEN}\,[\lambda.M.(1/p-1/2) + \lambda.Y.\alpha.\{-1/(h.\lambda)^2 + \lambda/12\}$$

$$+\frac{(a+b.n)}{h}\{\lambda.(1/p-1/2)\} + (1+\lambda.D)\frac{(a+b.n)}{-h^2}$$

$$-\frac{NUM}{DEN^2}\,[\,(1-\delta_1).\lambda.t_o.\alpha\ \{-1/(h^2.\lambda)+\lambda/12\}$$

$$+\lambda.(1/p-1/2)]$$

where,

$$NUM = \lambda.M.D + \lambda.B_o.Y + \lambda.W + \frac{(a+b.n)}{h}(1+\lambda.D)$$

$$DEN = (1+\lambda.B) + (1-\delta_1).\lambda.t_o.B_o$$

The equation $\partial c/\partial h = 0$ is solved for optimum n using Newton- Raphson's method starting from the initial root.

A computer program is developed in C language for the algorithm explained above. The program is written for the design of $\overline{X}$ chart and np chart. However, the program can be used for designing other charts by providing a function to calculate the average run-length based on the underlying distribution. After finding the design parameters, the program finds the optimum cost and compares it with the expected cost of user-defined parameters.

5. CASE STUDIES

The case studies were conducted in two modern iron foundries. In the first case, the study was carried out on the castings of sole plate telephone pole. In the second case, the product was cylinder liner. Economically optimum np-charts were designed for both the sole plate and cylinder liner. The input parameters were estimated from data. Some of the parameters were assumed:

Case 1

Process Parameters

Average production rate = 150 castings/hour

Cost of each casting = Re 0.75

Average in-control fraction defective, p_o = 0.0585

Shift in the process (in multiples of standard deviation), δ = 0.75

Out-of-control fraction defective,
$p_1 = p_o + \partial\sqrt{p_o(1-p_o)/n}$ = 0.0906

Penalty cost per hour = Production rate × unit cost × $(p_1\text{-}p_o)$ = Rs 361 per hour.

The in-control period = 30 hr

λ = 1/30 = 0.0333

Cost for locating and repairing the assignable cause, W = Rs 1000

Cost per false alarm, Y = Rs 500

Fixed cost per sample, a = Rs 10

Variable cost per unit sample, b = Rs 2

Search time for assignable cause, t_1 = 0.5 hr

Repair time, t_2 = 1 hr

Search time for false alarm, t_o = 0.5 hr

The time to sample and chart one item, g = 0

The process was allowed to continue during search and stopped during repair.

Hence $\partial_1 = 1$, $\partial_2 = 0$

By inputting the above parameters, the program gave the economically optimum design parameters of the control chart. The output of the program was

Sample size, $n = 28$

Width of control limits, $k = 2.14$

Sampling frequency, $h = 1.33$

The expected minimum cost per hour = Rs 175.33

With user defined design parameters ($n = 30$, $k = 3$, $h = 1$), the expected cost per hour = Rs 204.2. With economic design parameters, savings in the process control cost = Rs 28.68/hr.

Case 2

Process Parameters

$p_o = 0.0416$, $\partial = 0.75$, $\lambda = 0.03125$

$M = 307$, $W = 1000$, $Y = 500$

$a = 10$, $b = 0$, $t_1 = 4$, $t_2 = 2$, $t_o = 0$, $g = 0$,

$\partial_1 = 0$, $\partial_2 = 0$

With these process parameters, economic design parameters were:

Sample size, $n = 32$

Width of the control limits, $k = 2.41$

Sampling frequency, $h = 0.39$

The expected minimum cost/hour = Rs 78.75

With user-defined parameters ($n = 30$, $k = 3$, $h = 1$), the expected cost/hour = Rs 119.29

With economic design parameters, savings in the process control cost = Rs 40.54 /hr

5.1 Comparison with other Models

The derived economic model for the design of control charts was compared with some economic models which were developed on the basis of Duncan's guidelines. The drawbacks of the past models and the advantages of the suggested model over the past models are discussed.

5.1.1 Duncan's Single Cause Model

In 1956, Duncan[1] proposed an economic model for the design of control charts. His model dealt with a fully economic model of a control chart for the first time, and also incorporated formal optimisation methodology into determining the control chart parameters. Duncan assumed that the process was allowed to continue in operation during the search for an assignable cause, and the cost of eliminating the assignable cause was not charged against the net income for the period. The time to repair the assignable cause was also not considered in cycle time. In many processes, these restrictions are unrealistic.

The proposed model incorporates more flexibility. The model is applicable whether the process is stopped or continued during search or repair by considering appropriate terms for ∂_1 and ∂_2. Time for a false alarm is considered in cycle time. The repair time of the assignable cause is also considered in cycle time.

5.1.2 Montgomery's Model

Montgomery developed an economic model[4] for the design of $\bar{X}$ charts. One of the assumptions he considered was that the process was stopped while a search for an assignable cause was performed. He didn't mention whether the process was ceased or continued during repair.

5.1.3 Gibra's Model

Gibra presented a model for the economic design of attribute control charts for multiple assignable causes. He derived an equivalent matched single cause model[7]. He assumed that the process was stopped during search for an assignable cause. His model is more or less similar to Montgomery's model and is not suitable for continuous production.

5.1.4 Chiu's Model

Chiu developed an economic model[3] for the design of attribute control charts. The drawbacks of Chiu's model were :

(a) His model did not allow any time for taking a sample inspection and charting.

(b) The process stopped during search.

(c) The cost of repair was not considered in the cost function.

A model is being developed rectifying the above cited drawbacks.

6. SENSITIVITY ANALYSIS

Sensitivity analysis is the study of the effect of discrete parameter changes in the design. An economically optimum *np* chart was designed for the cast plate. Sensitivity analysis was performed to quantify the effect of process parameter changes on the design. Sensitivity analysis was performed for the following parameter changes:

(a) Shift, ∂ (as shift increases, penalty cost M also increases)

(b) Cost per false alarm, Y

(c) λ i.e., 1/mean in-control period.

It was found that the increase in process shift ∂ reduced optimum n, increased optimum k, and increased h initially and then reduced it. Increase in false alarm cost did not change n and k but increased h. Increase in λ did not change n and k but reduced h.

7. CONCLUSIONS

A general model has been developed for the economically optimum design of control charts. This model is applicable to all the control charts, regardless of the statistic used. It is only necessary to calculate the average run-length (α, β values) of the statistic assuming that the process is in-control and out- of-control in some specified manner. The general model has been developed rectifying the drawbacks of the past models. The cost function serves as a useful tool for quantifying process control costs and for evaluating changes in the fundamental process. An algorithm has been given to find economical design. Minimisation of the cost function over the choice of design parameters leads to most economical control chart. Considerable cost savings can be achieved without changing the fundamental control chart format.

Case studies were performed in two modern iron foundries. Economically optimum *np* control charts were designed for a cast plate and a cylinder liner. The sensitivity analysis was performed to quantify the effect of process parameter changes on the economic design. The economic design was relatively insensitive to errors in estimating the cost coefficients.

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