Defence Science Journal, Vol 48, No 4, October 1998, pp. 351-355

Surface Displacements under Harmonic Concentration

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ABSTRACT

Expressions are obtained for the surface displacements over a thermodiffusive elastic half space in which the concentration is assumed to be a harmonic function. As expected, these expressions are found to be extensions of the classical results. Some special cases are also included and various simpler expressions for the displacements are obtained, with plausible applications to defence science.

NOMENCLATURE

a	Compressional wave velocity
b	Shear wave velocity
C	Concentration at a subsequent time
C^*	Vibrational amplitude of C
d	Rate of change of chemical potential wrt absolute temperature at constant strain and concentration
d_c	Coefficient of linear diffusive expansion
d_t	Coefficient of linear thermal expansion
D_c	Fick's diffusion coefficient
F	Scalar displacement potential
F^*	Vibrational amplitude of F
G	Vector displacement potential
G^*	Vibrational amplitude of G
k	Coefficient of thermal conductivity
K	Bulk modulus
n	Wave number
s	Specific heat at constant strain
t	Time
T	Temperature perturbation of the absolute temperature from T_0
T_o	Initial reference temperature
T^*	Vibrational amplitude of T
u	Displacement vector
v	Velocity of propagated wave
w	Frequency

Greek Symbols

λ,μ	Lame constants
ρ	Density of the material
σ	Stress tensor
σ_{xy}	Tangential stress
σ_{yy}	Normal stress

Notations

$$a^2 = (\lambda + 2\mu) / \rho$$

$$b^2 = \mu / \rho$$

$$D_k = k + dT_o D_c$$

Received 29 July 1997, revised 29 January 1998

$f^2 = n^2 - w^2 / a^2$

$g^2 = n^2 - w^2 / b^2$

$\mathbf{G} = (0, 0, -G)$

$J = (2b^2 n^2 - w^2)^2 - 4fg n^2 b^4$

$J_1 = a(2b^2 - v^2)^2$

$\quad - 4b^3 [(a^2 - v^2)(b^2 - v^2)]^{\frac{1}{2}}$

$J_2 = \left[J_1 + av^2 (2b^2 \quad v \;) \right] / 2b^2$

$K = \lambda + (2/3)\mu$

$L = -(2b^2n^2 - w^2)\left[m_t A_o (p^2 - f \quad)^{-1} \right.$

$\left. \quad + m_c B_o (n^2 - f^2)^{-1} \right]$

$m_c = 3Kd_c / \rho a^2$

$m_t = 3Kd_t / \rho a^2$

$M \quad -2inb^2 \left[pm_t A_o (p^2 - f^2) \right.$

$\left. \quad + n\, m_c B_o (n^2 \quad f^2) \right]$

$p^2 = n^2 - iws / D_k$

$u = \text{grad } F + \text{curl } \mathbf{G}; \quad \text{div } \mathbf{G} = 0$ *or*

$u_x = F,_x - G,_y$

$u_y = F,_y + G,_x$

$u_z = 0$

with Commas indicating partial differentiation

$v = w / n$

1. INTRODUCTION

Over the years, multiple uses of diffusional problems in soil mechanics, defence science and polymers have been discovered. However, very few mathematical models are readily available to describe such realistic situations. This is mainly because the complex physical mechanisms involved are not yet settled by chemical engineers or material scientists[1,2]. However, there is consensus on two main points: (i) viscoelastic stresses play a major role in the diffusive patterns of common materials, and (ii) Fick's diffusion equation is not adequate to yield a model having properties that fit into a realistic situation. For example, to satisfactorily model the diffusion phenomena in semiconductors, one can use a nonlinear generalisation of the linear Fickian diffusion equation[3]. Irreversible processes like thermodiffusion can be described with the help of modern nonequilibrium thermodynamics[4]. Various similar theories of practical interest are also available in the literature[5-13].

In this paper, a new diffusion model based on thermoelasticity has been proposed, by considering a thermodiffusive elastic half space, initially maintained at a constant temperature and in which the concentration is assumed to be a harmonic function. It was observed that, in general, the diffusion of concentration was mainly due to the presence of pores or voids in the elastic material, that would help to smoothen the equilibrium solutions. Unlike hyperbolic problems, the parabolic problems smooth discontinuities in wave propagation[14] and because of this, a generalised version of the parabolic Fick's diffusion equation was used in the present study.

2. GOVERNING EQUATIONS

A rectangular cartesian coordinate system (x,y,z) is set up with the origin on the surface of the thermodiffusive elastic half space in such a way that the half space occupies the region $y \geq 0$ in R^3 and $y = 0$ corresponds to the free surface. The medium is initially maintained at a constant temperature T_o and the concentration is assumed to be a harmonic function. All the quantities considered are assumed to be independent of the z-coordinate so that the problem reduces to a two-dimensional plane strain problem. The governing equations are then given by

$$\nabla^2 F - \ddot{F} / a^2 = m_t T + m_c C$$
$$\nabla^2 G = \ddot{G} / b^2$$
$$D_k \nabla^2 T = s\dot{T}$$
$$\nabla^2 C = 0 \tag{1}$$

It was noted from Eqn (1) that F, T and C are coupled, while G remains uncoupled, thereby implying that shear waves are neither influenced by temperature nor concentration.

3. SOLUTIONS

To solve Eqn (1) the solutions are assumed as

$$(F, G, T, C) = (F^*, G^*, T^*, C^*) \exp[i(nx-wt)] \quad (2)$$

Eqn (1) then reduces to a set of ordinary differential equations which can be solved to obtain expressions for F^*, G^*, T^*, C^*. These are then used in Eqn (2), and keeping in mind that F, G, T, C describe surface waves, the following solutions are arrived at:

$$F = \left[F_o e^{-fy} + \frac{m_t A_o e^{-py}}{p^2 - f^2} + \frac{m_C B_o e^{-ny}}{n^2 - f^2}\right] \exp[i(nx - wt)]$$

$$G = G_0 \exp[-gy + i(nx-wt)]$$

$$T = A_0 \exp[-py + i(nx-wt)]$$

$$C = B_0 \exp[-ny + i(nx-wt)] \quad (3)$$

where $Re(f) \geq 0$, $Re(g) \geq 0$, $Re(p) \geq 0$, $Re(n) \geq 0$ and F_o, G_o, A_o, B_o are constants to be determined by imposing stress-free boundary conditions.

4. BOUNDARY CONDITIONS

Over the free surface $y = 0$, one sets:

$$\sigma_{yy} = 0 = \sigma_{xy}$$

and

$$(T,C) = (T_0, C_0) \exp[i(nx-wt)]\,;\, C_0 \neq 0 \quad (4)$$

The stress tensor is given by

$$\sigma = \mu(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) + [\lambda(\nabla \cdot \boldsymbol{u}) - 3K(d_t T + d_c C)]\boldsymbol{I}$$

which implies that

$$\sigma_{yy} = 2\rho b^2 (G_{,xy} - F_{,xx}) + \rho F_{,tt}$$

$$\sigma_{xy} = \rho b^2 (2F_{,xy} + G_{,xx} - G_{,yy}) \quad (5)$$

Eqns (3), (4) and (5) together yield

$$(2b^2n^2 - w^2)F_o - 2ingb^2 G_o = L$$

$$2infb^2 F_o + (2b^2n^2 - w^2)G_o = M \quad (6.1)$$

$$A_0 = T_0 \; ; \; B_0 = C_0 \quad (6.2)$$

which on solving will give

$$F_o = J^{-1}\left[(2b^2n^2 - w^2)\; L + 2ingb^2M\right]$$

$$G_o = J^{-1}\left[(2b^2n^2 - w^2)\; M - 2infb^2L\right] \quad (7)$$

In the absence of temperature and concentration, Eqn (6.1) is consistent and yields a non-trivial solution only when $J = 0$. It may be noted that $J = 0$ is equivalent to

$$\left[2 - \frac{v^2}{b^2}\right]^2 = 4\left[1 - \frac{v^2}{a^2}\right]^{\frac{1}{2}}\left[1 - \frac{v^2}{b^2}\right]^{\frac{1}{2}} \quad (8)$$

This is clearly the classical equation derived by Rayleigh for surface waves over an elastic half space, the boundary being stress-free[15-16].

5. DISPLACEMENTS

The displacements within the solid are given by

$$u_x = F_{,x} - G_{,y}$$

$$u_y = F_{,y} + G_{,x}$$

Eqn (3) will then yield

$$u_x = (g\, G_o e^{-gy} + inF_o e^{-fy})\exp[in(x-vt)$$
$$+\frac{ia^2}{v}\left[\frac{m_t A_o D_k e^{-py}}{nvD_k - isa^2} + \frac{m_C B_o e^{-ny}}{nv}\right]$$
$$\times \exp[in(x-vt)]$$
$$u_y = (-fF_o e^{-fy} + inG_o e^{-gy})\exp[in(x-vt)]$$
$$-\frac{a^2}{v}\left[\frac{m_t A_o D_k e^{-py}}{nvD_K - isa^2}\left(1-\frac{ivs}{nD_k}\right)^{\frac{1}{2}}\right.$$
$$\left.+\frac{m_C B_o e^{-ny}}{nv}\right]\exp[in(x-vt)] \quad (9)$$

5.1 Surface Displacements

The surface displacements over the free surface are obtained by putting $y = 0$ in Eqn (9). Further, making use of Eqn (7), one gets:

$$u_x = \frac{ia^2}{..r}\left[\frac{m_t T_o D_k}{\quad}\left[\right.\right.$$
$$+2abv^2(b^2-v^2)^{\frac{1}{2}}\left(1-\frac{ivs}{nD_k}\right)^{\frac{1}{2}}$$
$$+\frac{m_C C_o}{nv}\left[J_1 - (2b^2-v^2)J_2\right.$$
$$\left.\left.+2abv^2(b^2-v^2)^{\frac{1}{2}}\right]\right]\exp[in(x-vt)]$$

$$u_y = \frac{a^2}{vJ_1}\left[\frac{m_t T_o Dk}{nvD_k - isa^2}\left[(2b^2J_2 - J_1)\left(1-\frac{ivs}{nD_k}\right)^{\frac{1}{2}}\right.\right.$$
$$\left.-v^2(2b^2-v^2)(a^2-v^2)^{\frac{1}{2}}\right] + \frac{m_c C_o}{\quad}$$
$$\times\left[(2b^2J_2 - J_1 \quad -v^2(2b^2-v^2\right.$$
$$\left.\left.\times (a^2-v^2)^{\frac{1}{2}}\right]\right]\exp[in(x-vt)]$$

Eqn (10) gives the surface displacements over a thermodiffusive elastic half space, wherein the concentration is harmonic.

5.2 Case Studies

Case 1

In the absence of temperature and concentration, Eqn (9) will be reduced to the classical equations of Rayleigh.

Case 2

In the case of high frequencies, taking n also to be large such that $v = w/n$ remains finite, Eqn (10) is reduced to

$$u_x = \left[J_1 - (2b^2 - v^2 \quad J_2 + 2abv^2(b^2-v^2)^{\frac{1}{2}}\right]$$
$$\times \frac{ia^2}{vJ_1}\left[\frac{m_t T_o D_k}{nvD_k - isa^2} \quad \frac{m_c C_o}{\quad}\right]$$
$$\times \exp[in\ (x-vt)]$$

$$u_y = \left[(2b^2J_2 - J_1) - v^2(2b^2-v^2)(a^2-v^2)^{\frac{1}{2}}\right]$$
$$\times \frac{a^2}{vJ_1}\left[\frac{m_t T_o D_k}{nvD_k - isa^2} + \frac{m_c C_o}{nv}\right]$$
$$\times \exp[in\ (x-vt)]$$

Case 3

In case of small wave numbers, $v = w/n$ becomes large but the displacements become vanishingly small.

6. CONCLUSIONS

Expressions for the surface displacements over a thermodiffusive elastic half space are obtained. It is concluded that when the additional effects of temperature and concentration are neglected, the classical results of Rayleigh are reproduced. Various case studies clearly exhibit the effects of temperature and concentration on the displacements. Furthermore, the displacements become vanishingly small as the velocity increases.

ACKNOWLEDGEMENT

The author expresses her sincere thanks to the Department of Atomic Energy for financial support in the form of an *NBHM Research Award* and to Prof K.S. Harinath for guidance and encouragement.

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