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SHORT COMMUNICATION

Optimisation Problem of Entry into Earth's Atmosphere

S. K. Gurtu

Institute for Systems Studies & Analyses, Delhi-110 054

ABSTRACT

A study has been carried out on the variation of velocity, time, re-entry angle and distance along the horizontal with altitude for a re-entry vehicle diving into the earth's atmosphere, using the improved version of Marinescu's model that accounts for gravity and assuming that the distance along the earth's surface is fixed. More specifically, after formulating the problem as an isoperimetric one, its Euler-Lagrange equation, which turned out to be a highly nonlinear differential equation of the second order, has been solved via Runge-Kutta method and Simpson's rule for some physically realisable values of the parameters involved.

1. INTRODUCTION

The entry of a space vehicle into the earth's atmosphere is of interest to space scientists, planetologists and others. There are a number of satellites moving in near-earth orbit, the eventual entry of which into the earth's atmosphere poses wide ranging problems. Similarly, a large number of planetary probes during their final plunge radio back vital information of the planetary atmosphere, such as temperature, pressure, composition, etc. With the successful launching of space shuttle, a versatile hypervelocity vehicle capable of re-entering the earth's atmosphere, this problem has acquired additional significance. Besides, the path of an intercontinental ballistic missile (ICBM) in the dense atmospheric layers is pertinent to defence analysts and strategists. The optimisation of the entry of a space vehicle into the earth's atmosphere constitutes a problem which is different from the general one. The latter provides an idea of the variation of various trajectory parameters with time.

In this paper, it is intended to examine the minimum time model of Marinescu[1] more closely. Marinescu had considered re-entry under the influence of drag only and had preferred to ignore gravitational effects. Possibly, this was done for the sake of simplicity. However, in the present analysis, the effect of gravity has been considered and recourse has not been taken to any approximation. Further, the earth has been regarded as spherical and not flat.

2. EQUATIONS OF MOTION

The two-dimensional equations of motion[2], for the lifting vehicle entering the earth's atmosphere (Fig. 1), taking into account the facts stated above, are

$$m\dot{V} = -\frac{S_A C_x}{2}\rho V^2 + mg_e\left(\frac{R_e}{R_e + z}\right)\sin\theta \tag{1}$$

$$mV\dot{\theta} = -\frac{S_A C_x}{2}\rho V^2 + \left\{mg_e\left(\frac{R_e}{R_e + z}\right)^2 - \frac{mV^2}{R_e + z}\right\}\cos\theta \tag{2}$$

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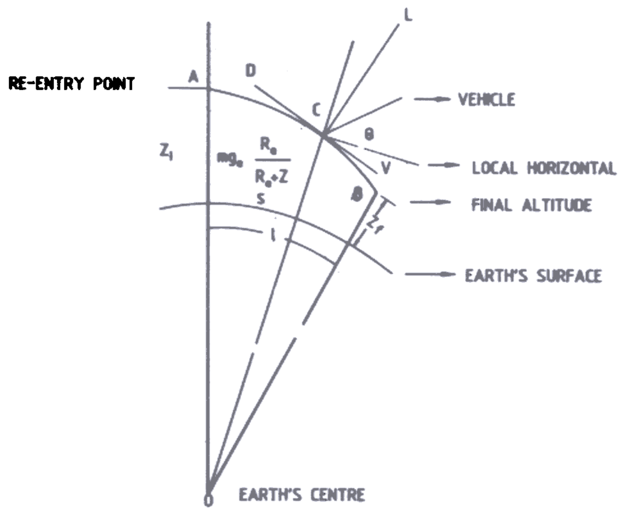

Figure 1. Geometry of the re-entry trajectory

$$\dot{z} = -V \sin\theta \tag{3}$$

$$\dot{S} = \left(\frac{R_e}{R_e + z}\right) \tag{4}$$

where m denotes mass of the vehicle; V, velocity; S_A, reference area; C_x, drag coefficient; C_z, lift coefficient; θ, inclination of the trajectory to the horizontal; z, altitude, and S, distance along the surface of the earth. The dots over V, θ, z and S represent differentiation wrt time.

As the variation of altitude in the problem is considerable (from 110 km at re-entry to 50 km finally) in Eqns (1) and (2), the force of gravity is not taken as a constant mg, where g is the acceleration due to gravity, but one depending on altitude, as follows:

$$mg = mg_e \left(\frac{R_e}{R_e + z}\right)^2 \tag{5}$$

where g_e is acceleration due to gravity at the surface of the spherical earth having radius R_e = 6,371,200 m. The dependence of g_e on the latitude of the place can be ignored.

The atmospheric density depends on altitiude and follows an exponential decrease[3].

$$\rho = \rho_o \exp(-\beta z) \tag{6}$$

where the density at sea level, $\rho_o = 1.225$ kg/m^3 and $\beta = 0.000395 \times m^{-1}$. However, Eqn (6) is not strictly $\dot{x} = V\cos\theta$, valid at higher altitudes, where it would be more appropriate to use the Standard ARDC Atmosphere (1959) data.

It may be noted that Eqn (4) of Marinescu is $\dot{x} = V\cos\theta$ which is valid for flat earth. In the present analysis, the equation is replaced by $\dot{S} = \left(\frac{R_e}{R_e + z}\right) V\cos\theta$, to take into account the earth's curvature.

Reverting to the differential equations, if

$$V' = \frac{dV}{dz}$$

$$a = \frac{S_A C_x}{2m}$$

and

$$b = g_e \left(\frac{R_e}{R_e + z}\right)^2$$

then

$$\sin\theta = \frac{a\rho V^2}{VV' + b} \tag{7}$$

For the lifting entry vehicle, the time required to travel the descending path between initial altitude, z_i and final altitude, z_f is

$$t = \int_{z_f}^{z_i} \frac{(VV' + b)dz}{a\rho V^3} \tag{8}$$

Using Eqns (3), (4) and (7), one obtains

$$S = \int_{z_f}^{z_i} \frac{[(VV' + b)^2 - (a\rho V^2)^2]^{1/2}}{a\rho V^2} dz \tag{9}$$

It may be pointed that at this stage, Marinescu deviated and instead of making use of the equation

$$x = \int_{z_f}^{z_i} \frac{(V'^2 - a^2\rho^2V^2)^{1/2}}{a\rho V} dz \tag{10}$$

took recourse [1] to the following approximation

$$x = \int_{z_f}^{z_i} \left(\frac{}{a\rho V / V'} \quad \frac{a\rho V}{2V'} \right) dz \tag{11}$$

The Eqn (11) is valid for small values of $a\rho V / V'$. There seems little justification for using Eqn (11), except perhaps that it simplifies calculations. Therefore, such an artifice will not be resorted to and Eqn (9) , which is a variant of Eqn (10), will be used without change.

The variational problem[4] consists in solving the minimum of the functional Eqn (8) given the isoperimetric condition:

$$\int_{z_f}^{z_i} \left(\frac{R_e}{R_e + z} \right) \frac{[(VV'+b)^2 - (a\rho V^2)^2]^{1/2}}{a\rho V^2} dz = l \tag{12}$$

The curve that achieves the extremum of the functional Eqn (8) is an extremum of the functional

$$J = \int_{z_f}^{z_i} H\, dz \tag{13}$$

where

$$H = \frac{VV' + b}{a\rho V^3} + \lambda \left(\frac{R_e}{R_e + z} \right) \frac{[(VV' + b)^2 - (a\rho V^2)^2]^{1/2}}{a\rho V^2} \tag{14}$$

Making use of Euler's equation

$$H_V - \frac{d}{dz} H_{V'} = 0 \tag{15}$$

one gets the differential equation, after simplification, as

$$\begin{aligned} V'' = & \frac{3bA^{3/2}}{\lambda \left(\frac{R_e}{R_e + z} \right) a^2\rho^2V^8} + \frac{2A^2}{a^2\rho^2V^7} \quad \frac{(VV' + b)V'A}{a^2\rho^2V^6} \\ & + \frac{2A}{V^3} + \frac{\beta A^{3/2}}{\lambda \left(\frac{R_e}{R_e + z} \right) a^2\rho^2V^6} + \frac{2(VV' + b)^2 b}{(R_e + z)a^2\rho^2V^5} \\ & - \frac{\beta(VV' + b)}{V} - \frac{2bA}{(R_e + z)a^2\rho^2V^5} + \frac{\beta(VV' + b)A}{a^2\rho^2V^5} \\ & - \frac{(VV' + b)A}{(R_e + z)a^2\rho^2V^5} - \frac{V'^2(VV' + b)^2}{a^2\rho^2V^5} \\ & + \frac{2(VV' + b)V'}{V^2} - \frac{V'bA}{a^2\rho^2V^6} \end{aligned} \tag{16}$$

where

$$A = (VV' + b)^2 - (a\rho V^2)^2$$

For the sake of comparison, when instead of taking a spherical earth, the simpler case of a flat earth is assumed, the differential Eqn (16) becomes

$$\begin{aligned} V'' = & \frac{3bA^{3/2}}{\lambda a^2\rho^2V^8} + \frac{2A^2}{a^2\rho^2V^7} - \frac{(VV' + b)V'A}{a^2\rho^2V^6} \\ & + \frac{2A}{V^3} + \frac{\beta A^{3/2}}{\lambda a^2\rho^2V^6} + \frac{2(VV' + b)^2 b}{(R_e + z)a^2\rho^2V^5} \\ & \frac{\beta(VV' + b)}{V} - \frac{2bA}{(R_e + z)a^2\rho^2V^5} + \frac{\beta(VV' + b)A}{a^2\rho^2V^5} \\ & \frac{V'^2(VV' + b)^2}{} + \frac{2(VV' + b)V'}{V^2} - \frac{V'bA}{a^2\rho^2V^6} \end{aligned}$$

It may be noted that the difference between differential Eqns (16) and (18) is that in the former more general Eqn (16) $\lambda\left(\frac{R_e}{R_e + z}\right)$ is used for λ. Also, differential Eqn (16) contains an additional term $-\frac{(VV' + b)A}{a^2\rho^2V^5(R_e + z)}$.

Finally, for the still simpler case of flat earth and gravityless condition Eqn (16) takes the form

$$V'' = \frac{\beta(V'^2 - a^2\rho^2V^2)^{3/2}}{\lambda a^2\rho^2V^3} + \frac{V'^3\beta}{a^2\rho^2V^2} - 2\beta V' + \frac{V'^2}{V} \tag{19}$$

which follows, as expected, by putting $b = 0$ in differential Eqn (17).

3. NUMERICAL SOLUTIONS

The second order differential Eqns (16), (18) and (19) have been solved by Runge-Kutta[5] method for a re-entry vehicle having V_i = 7.8 km/s, z_i = 110 km, z_f = 50 km, $\theta_i = 3°$, a = 0 .005 and l = 400 km.

For the three different cases, corresponding to l = 400 km(fixed), the variation of velocity with altitude is shown in curves A, B and C (Fig. 2). The variation of velocity was initially slow but became fast at lower altitudes. For curves A and B it was noticed that immediately after re-entry, the velocity of the vehicle increased till an altitude of about 92 km was reached. Thereafter, it decreased continuously. The initial increase in velocity was due to the fact that gravity term dominates the drag term and only later on, the effect of the denser atmosphere starts exerting itself and results in decrease in velocity. In case of curve C: a continuous decrease is noticed throughout the descent phase.

In the curves D, E and F (Fig. 2), the variation of time with altitude is shown. In this case, the integral given by Eqn (8) is solved by Simpson's rule. The maximum spread is in the region around 90 km initially and at lower altitudes, the curves tend to merge. The variation of angle with altitude is shown in curves A, B and C (Fig. 3). For curves A and B, it was noticed that the angle decreased

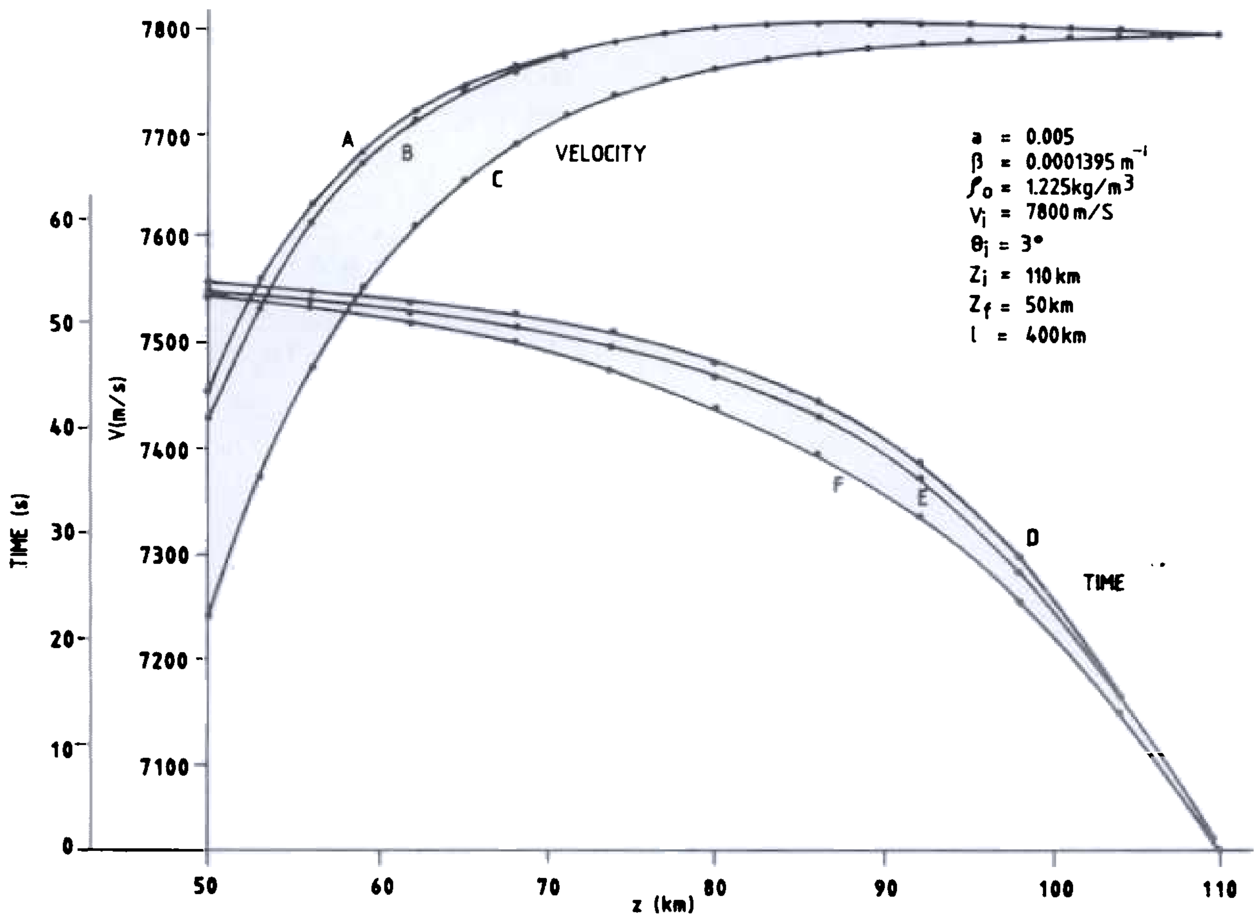

Figure 2. Variation of velocity and time with altitude for minimum time re-entry trajectory (A & D-gravity plus spherical earth; B&E-gravity plus flat earth and C & F-gravity neglected plus flat earth.

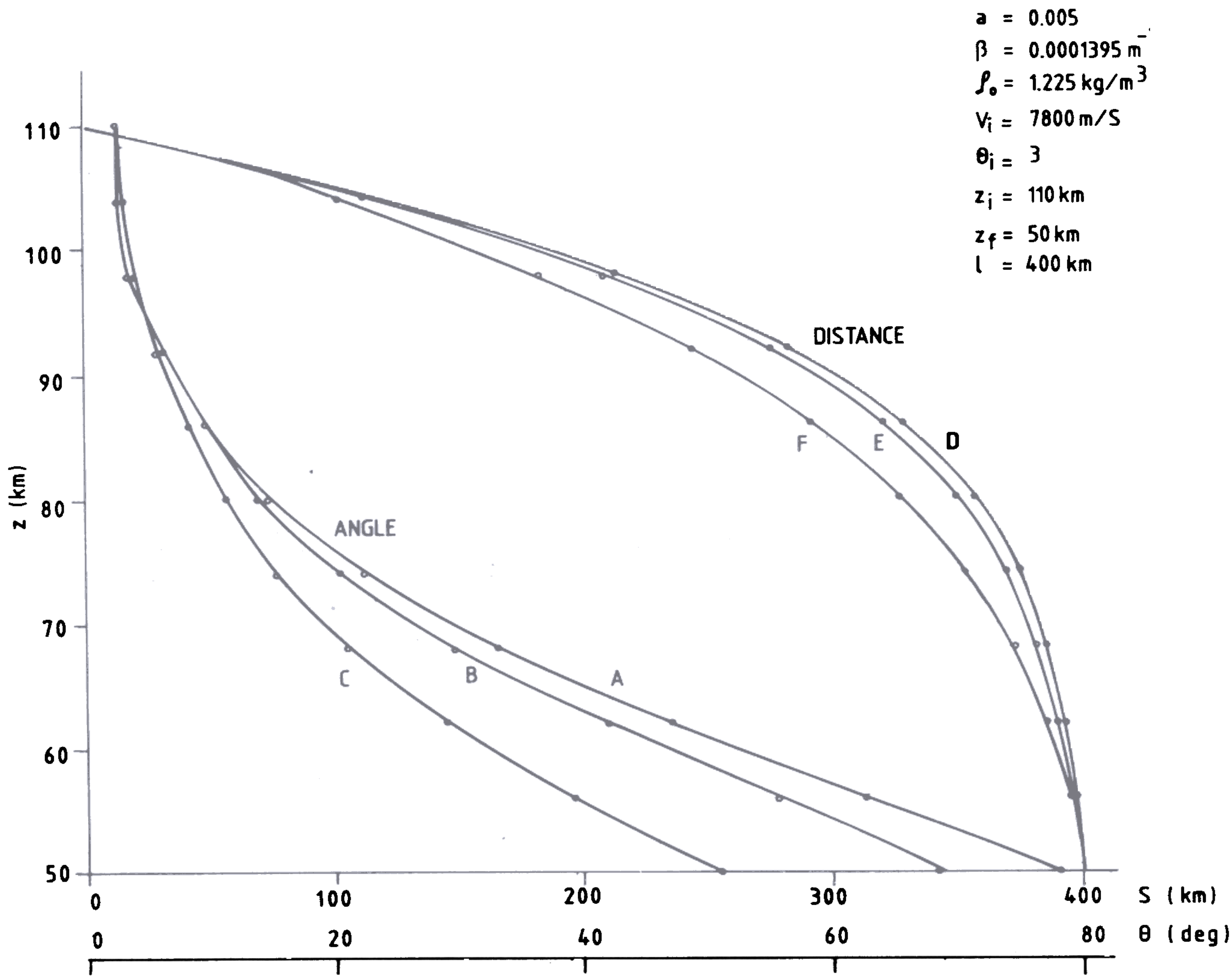

Figure 3. Variation of angle and distance along the horizontal with altitude for minimum time re-entry trajectory (A & D-gravity plus spherical earth; B & E-gravity plus flat earth and C&F-gravity neglected plus flat earth.

initially but subsequently it increased. In case of curve C, a continuous increase in the angle was noticed. The variation of horizontal distance with altitude is shown in curves D, E and F. The maximum spread was around an altitude of 90 km. It is apparent that curves A and B (Figs. 2 and 3) are more akin to one another as compared to curve C. Similarly, curve D and E are close to one another as compared to F.

4. DISCUSSIONS & CONCLUSIONS

The solution of differential Eqns (16), (18) and (19) indeed provides the minimum of the functional Eqn (8) because Legendre's condition $H_{\gamma'\gamma'} > 0$ for a weak minimum is satisfied. It may be mentioned that if the initial values and constants are known then the differential Eqns (1) to (4) can be solved. In case of minimum time problem, one has to choose a value of C_2 such that the variation of parameters V, z, θ, and S with time matches with what has been obtained earlier.

The minimum time problem cannot be solved by taking arbitrary values. This fact has been pointed out in our earlier communication[6]. For l = 40 km (say) cannot be obtained because the present re-entry angle is very small. Probably a very high re-entry angle will be required to achieve the objective. This situation does not have much physical relevance because it produces excessive heating. Similarly, a very high value of l = 4000 km may not be realisable as very low re-entry angle is required and in that case of shallow entry, the vehicle may simply bounce back in deep space.

The present study may have importance in planetary entry because of large variations expected in surface gravity and atmospheric density. In the present model, the atmosphere has been considered as stationary, i.e,. wind effect has been ignored. It would be interesting to study the role of head/tail winds. Besides studying the minimum time problem, Marinescu also dealt with the total minimum heat input case. In the same communication, he gave allusions to two more cases, viz; minimum consumption of ablative mass and minimum heat yielded in the critical zone. Besides minimising time, some other parameter, say, the horizontal distance can be minimised. The numerical calculations were carried out using a PC.

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Contributor

Dr SK Gurtu obtained his MSc in Mathematics from Agra University in 1963 and DPhil from Allahabad University in 1972. He is a member of the National Academy of Sciences, Allahabad and a member of the Operations Research Society of India, Delhi. His areas of research include astrophysical, astronautical and re-entry problems. He has also worked on application of OR/SA techniques to naval problems and in developing mathematical models. Presently, he is working on problems concerning dynamical oceanography.