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Applications of a Nonlinear Constitutive Equation for Creeping Snow

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ABSTRACT

In the seventies the author carried out numerous laboratory tests, simultaneously performed under six different states of stress and deformation (totally 121 identical three samples tested). The aim was to obtain a three dimensional nonlinear constitutive equation, i.e., one which higher applies to stresses.The theoretical background was a constitutive equation consistent with the principle of maximum entropy production. The irreversible part – which was exclusively considered – depends only on the dissipation function, represented by an exponential series. The final result consists of nine coefficients of three invariants of the stress tensor. Unfortunately, the resulting equation was never used to resolve practical problems in snow mechanics. This paper is aimed to demonstrate the usefulness of the equation by means of simple examples. For a uniform horizontal snow cover, it was firstly shown that snow behaves strongly non-symmetrically under compression and tension. And secondly, it was seen that the settlement (compression) deformation rates are up to 50 per cent higher than those with linear behaviour. In an other example, the development of a shear crack on the occasion of snow slab release has demonstrated that fracture starts earlier upslope and propagates faster than downslope. On the other hand, linearity between shear stresses and shear deformation can be justified.

1. INTRODUCTION

Solutions of problems in applied snow mechanics include the calculation of the velocity and stress fields. Besides the equations which hold for any continuum – equations of continuity and motion – further equations which describe the specific material in question are needed. These are the constitutive equations establishing a relation between statical (stress) and kinematical (rates of deformation) tensors.

A constitutive equation has been developed by Salm[3,4] based on numerous laboratory tests (uniaxial stress with compression and tension, uniaxial deformation, triaxial tests, i.e., uniaxial compressive and tensile stresses with superimposed hydrostatic pressure and finally pure hydrostatic pressure. All tests under different states of stress and deformation were performed simultaneously with the same snow type under same temperature conditions.

Unfortunately, the results of these extremely time-consuming and expensive investigations were never used in practice. One of the probable reasons for this was that these were presented in the form of an unpublished dissertation in German only.

The idealisations made were: snow samples are an isotropic and homogeneous continuum. Temperature remains always constant. Creeping means slow movements with no inertial effects. Furthermore elasticity including retarded elasticity are not considered, i.e., only the states after retardation

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time has passed by are considered.

The resulting constitutive equation holds only for a momentary state of the snow with a certain momentary structure (coordination number and area of bonds between grains, grain shapes and diameters, no influence of metamorphism, etc.). The experiments showed that although their durations were only a few hours, a surprisingly large influence of even small deformations (a few per cent of change in length of the sample produced a change in viscosity of one order of magnitude larger. So, as the snow after the test was no more the same as before – always a reference-state corresponding to the undeformed sample, had to be calculated. With regard to these idealisations the most general constitutive equation is that of a Reiner–Rivlin–liquid[2].

However with the principle of maximum entropy production postulated by Ziegler[6], the field of possible constitutive equations is substantially reduced. The irreversible part of deformation depends only on the dissipation function D:

$$V_{ij} = \rho D\left(\frac{\partial D}{\partial \sigma_{kl}}\sigma_{kl}\right)^{-1}\frac{\partial D}{\partial \sigma_{ij}} \quad (1)$$

where V_{ij} and σ_{ij} represent the tensor of deformation rate [s^{-1}] and stress [N/m^2] or [Pa], respectively.

$$D(\sigma_{ij}) = \frac{1}{\rho}\sigma_{ij}V_{ij} = 0 \quad (2)$$

where ρ is the mean density. The summation convention is used.

With the assumed isotropy, the right hand side of Eqn (1) can be expressed as Eqns (3) and (4)

$$\frac{\partial D}{\partial \sigma_{kl}}\sigma_{kl} = \frac{\partial D}{\partial \sigma_{(1)}}\sigma_{(1)} + 2\frac{\partial D}{\partial \sigma_{(2)}}\sigma_{(2)} + 3\frac{\partial D}{\partial \sigma_{(3)}}\sigma_{(3)} \quad (3)$$

and

$$\frac{\partial D}{\partial \sigma_{ij}} = \frac{\partial D}{\partial \sigma_{(1)}}\frac{\partial \sigma_{(1)}}{\partial \sigma_{ij}} + \frac{\partial D}{\partial \sigma_{(2)}}\frac{\partial \sigma_{(2)}}{\partial \sigma_{ij}} + \frac{\partial D}{\partial \sigma_{(3)}}\frac{\partial \sigma_{(3)}}{\partial \sigma_{ij}} \quad (4)$$

where

$$\frac{\partial \sigma_{(1)}}{\sigma_{ij}} = \delta_{ij} \quad (5)$$

$$\frac{\partial \sigma_{(2)}}{\sigma_{ij}} = \sigma_{ij} - \sigma_{(1)}\delta_{ij} \quad (6)$$

$$\frac{\partial \sigma_{(3)}}{\sigma_{ij}} = \sigma_{ik}\sigma_{kj} - \sigma_{(1)}\sigma_{ij} - \sigma_{(2)}\delta_{ij} \quad (7)$$

In these equations, $\sigma_{(1)}$, $\sigma_{(2)}$ and $\sigma_{(3)}$ are the invariants of the stress tensor and δ_{ij} is the Kronecker delta.

The results of the tests consisted of determination of the dissipation function. This function was represented by an exponential series in the invariants of the stress tensor. The convergence of the series had always to be checked. Its lowest degree is

$$\rho D = \alpha_{11}\sigma_{(1)}^2 + \alpha_{(2)}\sigma_{(2)} \quad (8)$$

and represents the Newtonian liquid and is denoted as linear behaviour. The nonlinear behaviour comes into play with degrees higher than two in stresses, e.g.,

$\alpha_{111}\ \sigma\ (1)^3$, etc.

Based on the tests, the α values could be determined including the uppermost necessary degree. The final dissipation function has the form:

$$\begin{aligned}\rho D = {} & \alpha_{11}\sigma_{(1)}^2 + \alpha_{111}\sigma_{(1)}^3 + \alpha_{1111}\sigma_{(1)}^4 + \\ & \alpha_{12}\sigma_{(1)}\sigma_{(2)} + \alpha_{112}\sigma_{(1)}^2\sigma_{(2)} + \alpha_2\sigma_{(2)} + \\ & \alpha_{22}\sigma_{(2)}^2 + \alpha_{222}\sigma_{(2)}^3 + \alpha_3\sigma_{(3)}\end{aligned} \quad (9)$$

The nine coefficients α are:

$$\left.\begin{aligned}
\alpha_{11} &= 5.320\times10^{-12} & [m^2s^{-1}N^{-1}] \\
\alpha_{111} &= -6.997\times10^{-17} & [m^2s^{-1}N^{-1}] \\
\alpha_{1111} &= 8.097\times10^{-22} & [m^2s^{-1}N^{-1}] \\
\alpha_{12} &= -2.324\times10^{-16} & [m^2s^{-1}N^{-1}] \\
\alpha_{112} &= 3.626\times10^{-21} & [m^2s^{-1}N^{-1}] \\
\alpha_{2} &= 1.363\times10^{-11} & [m^2s^{-1}N^{-1}] \\
\alpha_{22} &= 4.409\times10^{-21} & [m^2s^{-1}N^{-1}] \\
\alpha_{222} &= 3.929\times10^{-32} & [m^2s^{-1}N^{-1}] \\
\alpha_{3} &= -5.388\times10^{-16} & [m^2s^{-1}N^{-1}]
\end{aligned}\right\} \quad (10)$$

These α values are valid for the tested snow, i.e., for a density $\rho = 431.9$ kg m^{-3}, stresses within about $|0.7 \times 10^5 N\ m^{-2}|$ at -5.2 °C. Tensile strength was 2×10^5 N m^{-2} and the mean grain diameter 0.15 mm (rounded grains).

2. APPLICATIONS

Applications of the constitutive equation described above are possible for any three dimensional state of stress or deformation. The aim of this paper is, however, to demonstrate the character of the presented constitutive equation by simple one- dimensional state and to demonstrate some limits of the often used linear relationships.

2.1 Uniaxial State of Deformation

Under a stress σ_{11} (in direction x_1 or simply x), lateral stresses (σ_{22}) impede lateral rates of deformation $V_{22} = V_{33}$ perpendicular to direction x_1. In the principal axes x_1 x_2 and x_3, shear stresses (σ_{12}, σ_{13} and σ_{23}) and shear deformation rates (V_{12} V_{13} V_{23}) vanish.

From Eqn (1), it follows for i=j=2 that

$$\frac{\partial D}{\partial \sigma_{22}} = \frac{\partial D}{\partial \sigma_{33}} = 0 \tag{11}$$

If one sets

$$\frac{\sigma_{22}}{\sigma_{11}} = \frac{\sigma_{33}}{\sigma_{11}} = \lambda \tag{12}$$

the stress invariants become

$$\sigma_{(1)} = \sigma_{11}(1+2\lambda) \tag{13}$$

$$\sigma_{(2)} = -\sigma_{11}^2(2\lambda + \lambda^2) \tag{14}$$

$$\sigma_{(3)} = \sigma_{11}^3\lambda^2 \tag{15}$$

and the condition for lateral confined movements is:

$$V_{22} = V_{33} = 0 = \frac{\partial D}{\partial \sigma_{(1)}} + \frac{\partial D}{\partial \sigma_{(3)}}\left(-\sigma_{11}(1+\lambda) + \frac{\partial D}{\partial \sigma_{(3)}}\sigma_{11}^2\lambda\right) \tag{16}$$

In Eqn (16) the quantity in demand is λ, which can easily be determined numerically.

With known λ and setting i=j=1, V_{11} can be calculated by Eqn (1), where

$$\frac{\partial D}{\partial \sigma_{11}} = \frac{\partial D}{\partial \sigma_{(1)}} + \frac{\partial D}{\partial \sigma_{(2)}}(-\sigma_{11}2\lambda) + \frac{\partial D}{\partial \sigma_{(3)}}\sigma_{11}^3\lambda^2 \tag{17}$$

and

$$\frac{\partial D}{\partial \sigma_{kl}}\sigma_{kl} = \frac{\partial D}{\partial \sigma_{(1)}}\sigma_{11}(1+2\lambda) + 2\frac{\partial D}{\partial \sigma_{(2)}}\left(-\sigma_{11}^2(2\lambda+\lambda^2)\right) + 3\frac{\partial D}{\partial \sigma_{(3)}}\sigma_{11}^3\lambda^2 \tag{18}$$

For the dissipation function D and its derivatives wrt the invariants, similar expressions in σ_{11} and λ can be found by using Eqn (9).

For linear behaviour, λ and V_{11} can directly be calculated as:

$$\lambda = \frac{2\alpha_{11} - \alpha_2}{\alpha_2 - 4\alpha_{11}} \tag{19}$$

and

$$V_{11} = \left[\alpha_{11} + \lambda(2\alpha_{11} - \alpha_2)\right]\sigma_{11} \tag{20}$$

With the values given in Eqn (10), λ is positive and $(2\alpha_{11} - \alpha_2)$ is negative, as expected.

In linear snow mechanics, it is common to use the more intuitive magnitudes m (inverse of the viscous analogue of Poisson`s ratio) and η (viscosity) instead of α_{11} and α_2. The relations mentioned above can be rewritten as:

$$\lambda = \frac{1}{m-1} \tag{21}$$

and

$$V_{11} = \frac{m-2}{\eta 2(m-1)}\sigma_{11} \tag{22}$$

where

$$m = \frac{-2\alpha_{11}}{2\alpha_{11} - \alpha_2} \qquad (2 \le m \le \infty) \tag{23}$$

and

$$\eta = \frac{1}{\alpha_2} \tag{24}$$

are constants in contrast to the nonlinear behaviour.

2.2 Uniaxial State of Stress

In this state of stress, lateral stresses σ_{22} and σ_{33} vanish and, therefore, $\lambda = 0$.

For $i=j=1$ Eqn (1) yields

$$V_{11} = \alpha_{11}\sigma_{11} + \alpha_{111}\sigma_{11}^2 + \alpha_{1111}\sigma_{11}^3 \tag{25}$$

and with $i=j=2$ and $D = \sigma_{11}\ V_{11}$ one gets

$$\frac{V_{22}}{V_{11}} = 1 - \frac{\alpha_2 + \alpha_{12}\sigma_{11} + \alpha_{112}\sigma_{11}^2}{2\alpha_{11} + 3\alpha_{111}\sigma_{11} + 4\alpha_{1111}\sigma_{11}^2} \tag{26}$$

Equation (25) demonstrates that the rate of deformation is different under compression (σ_{11} = negative) and tension (σ_{11}= positive), which of course is an important result of the laboratory tests. Eqn (26) is identical to $-m^{-1}$ and shows that m is different in compression and tension and, therefore, also the rate of volume change.

For linear behaviour Eqns (25) and (26) yield

$$V_{11} = \alpha_{11}\sigma_{11} = \frac{1}{\eta}\frac{m}{2(m+1)}\sigma_{11} \tag{27}$$

and

$$\frac{V_{22}}{V_{11}} = 1 - \frac{\alpha_2}{2\alpha_{11}} = -\frac{1}{m} \tag{28}$$

where η and m are same as in Eqns (23) and (24).

3. APPLICATIONS FOR α VALUES

3.1 Uniform Horizontal Snowcover

In Fig.1 rates of deformation under compression V_{11} are plotted for a snowpack with a thickness of D = 5 m. The origin of coordinate x is at the surface. For the uniaxial state of deformation (settlement of the snowpack) deformation rates are much higher compared to the linear behaviour (+50 per cent at the bottom), whereas the λ values are smaller than the (constant) linear value of 0.391.

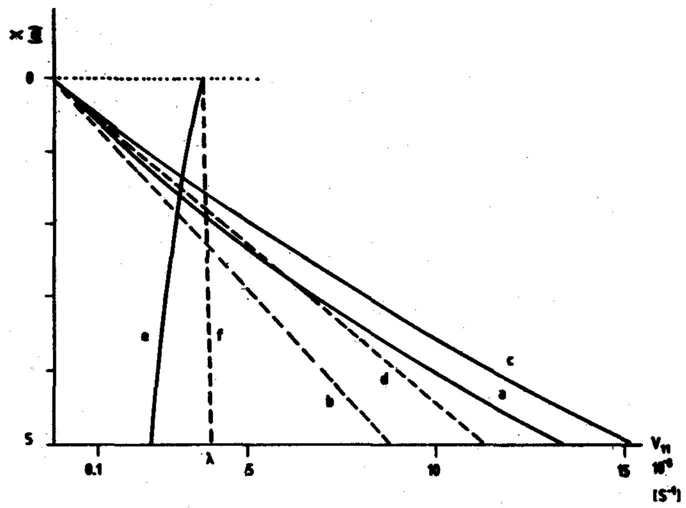

Figure 1. Deformation rates in a uniform horizontal snowcover under compression. Uniaxial state of deformation: *a*-nonlinear, *b*-linear; uniaxial state of stress: *c*-nonlinear, *d*-linear; λ values: *e*-nonlinear, *f*-linear.

For comparison, the uniaxial state of stress (λ = 0) is plotted too. Here again, considerable differences are observed (+35 per cent).

In Fig.(2) for comparison the same snowpack is exposed to tensile stresses. Again, considerable differences occur between linear and nonlinear behaviour (uniaxial states of deformation and of

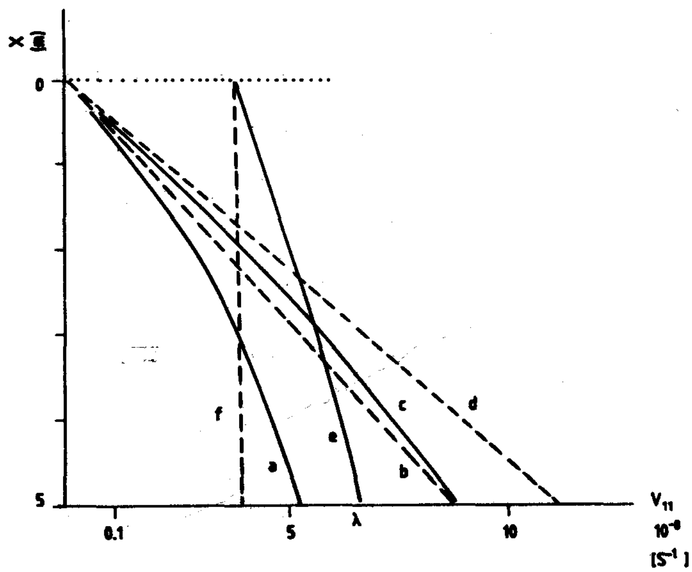

Figure 2. Deformation rates in a uniform horizontal snowcover under tension (imaginary). Uniaxial state of deformation: *a*-nonlinear, *b*-linear; uniaxial state of stress: *c*-nonlinear, *d*-linear; λ values: *e*-nonlinear, *f*-linear.

stress). The λ values are much higher than in the linear case (with 0.391).

3.2 Creep Velocities

To get the velocities u [$m\ s^{-1}$] in x-direction, one has to integrate V_{11} over D by using Eqn (1) and setting i=j=1 as described above. Density is ρ and g is the acceleration due to gravity. For linear behaviour this leads to the well-known analytical formulae for uniaxial state of deformation:

$$u = \rho g \left[\frac{1}{\eta} \frac{m-2}{2(m-1)} \right] \frac{D^2}{2} \left(1 - \frac{x^2}{D^2} \right) \quad (29)$$

and for uniaxial state of stress

$$u = \rho g \left[\frac{1}{\eta} \frac{m}{2(m+1)} \right] \frac{D^2}{2} \left(1 - \frac{x^2}{D^2} \right) \quad (30)$$

Figures 3 and 4 represent u for the uniaxial state of deformation under compression (settlement) and under tension, respectively. Again, both are compared with linear behaviour. As expected, large differences occur: snow is more soft under compression and more rigid under tension.

3.3 Shear Crack in Weak Layer

Bader and Salm[1] proposed a model for snow slab release by postulating the necessary existence of a slope-parallel super-weak zone which interrupts a weak layer over a certain distance 2α. It was assumed that shear stresses of the overburden snow layer cannot be transmitted over 2α. At the upper

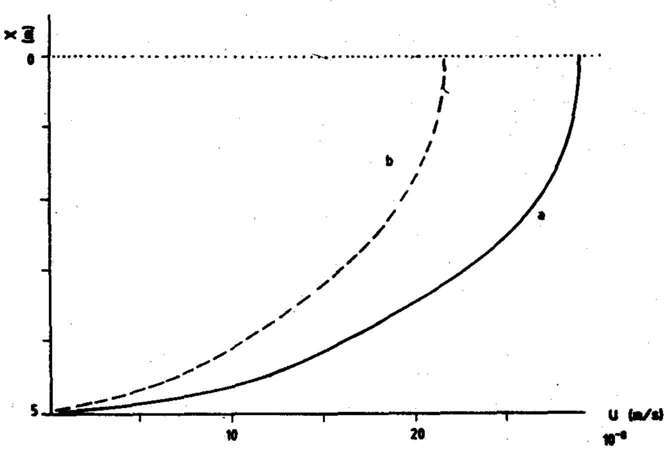

Figure 3. Creep velocities in a uniform horizontal snowcover under compression (settlement).

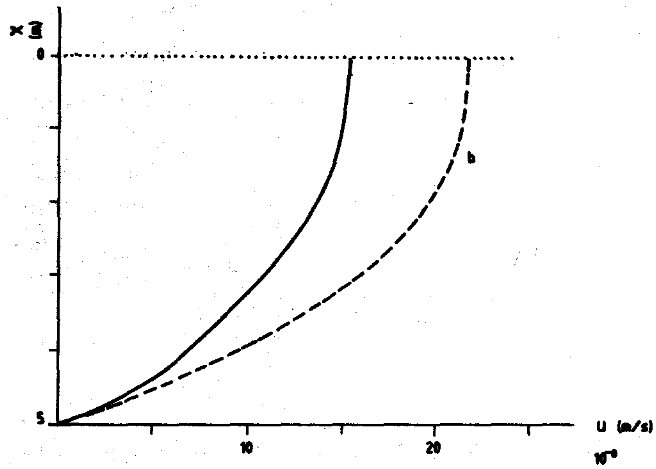

Figure 4. Creep velocities in a uniform horizontal snowcover under tension (imaginary).

end of the super-weak zone which is assumed to be infinite along the contour-line, additional tensile stresses appear in the overburden layer in a plane perpendicular to the weak layer. On the other hand, additional compressive forces occur at the lower end. These additional forces create additional shear stresses in the weak layer with their maximum at the ends of the super-weak zone and decreasing exponentially to zero with increasing distance from them. Superimposed are the shear stresses of the undisturbed snowpack, i.e.,

$$\sigma_{21}^{(0)} = \rho_0 g d_0 \sin(\psi) \quad (31)$$

where ρ_0 and d_0 are the density and the thickness of the overburden layer, respectively. Slope angle is ψ and axis parallel to the weak layer is x_1 and perpendicular to it x_2. The resulting maximum shear stress is:

$$\sigma_{21}^{max} = \sigma_{21}^{(0)} \left(\alpha_0 \frac{a}{d_0} + 1 \right) \quad (32)$$

where

$$\alpha_0 = \sqrt{\frac{(m-2)\eta_s d_0}{2(m-1)\eta_0 d_s}} \quad (33)$$

with the viscosities η_s and η_0 of the weak layer (thickness d_s) and the overburden layer respectively.

Bader and Salm assumed linear behaviour, therefore, the lengths of the upslope tension zone and compression zone are equal.

With the nonlinear constitutive equation however, the tension zone must be longer. To investigate this effect, the velocity distribution of the overburden layer above the super-weak zone was calculated. For this, spatially fixed planes perpendicular to the weak layer on both ends of the zone were assumed to have identical velocities on both ends. The quantity $\rho_0 \gamma \delta_0 \sin\psi$ of the overburden layer was taken as 4,237 Nm^{-2}. The total length of the zone is assumed to be 10 m . As boundary condition one has vanishing u at both the ends. The uniaxial state of deformation was used for calculation which is an approximation. In reality, the mechanical state is between uniaxial state of stress and deformation. The boundary between tension and compression zones is given with the condition $V_{11}= 0$, where stresses are zero. In the velocity distribution this is

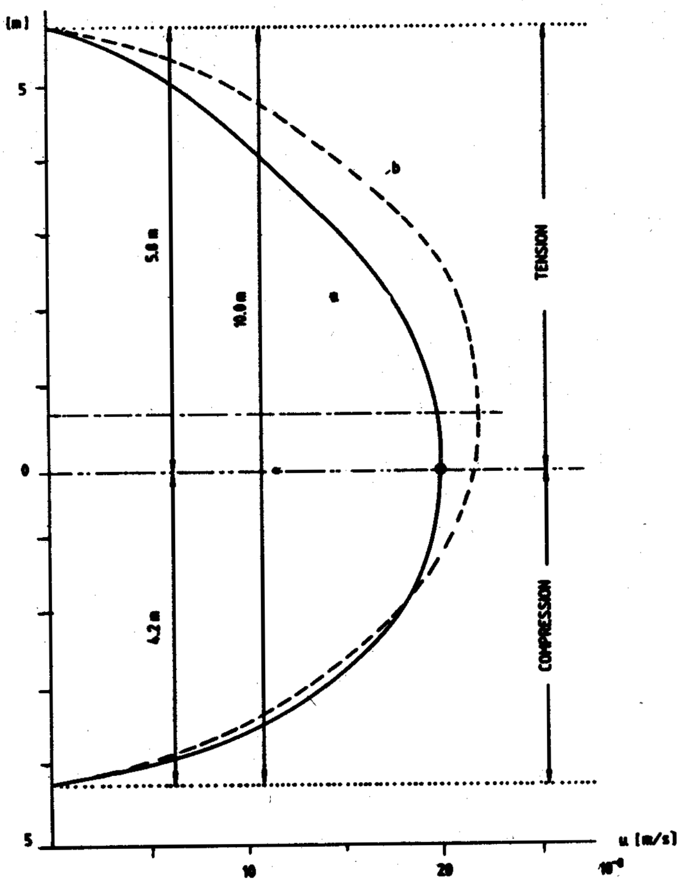

Figure 5. Creep velocitiy distribution of the over burden snow layer within the super-weak zone (*a*-nonlinear, *b*-linear)

given by the maximum, i.e., $\delta u/\delta x = 0$.

Figure 5 shows a 5.8 m tension zone (instead of 5 m in the linear case). Therefore in Eqn (32) one has to replace a by 1.16 a and 0.84 a the upper and the lower shear stress peaks, respectively.

In view of the uncertainity of η_s, this difference can certainly be neglected. It is however, an interesting consequence of snow slab release: upslope fracture in the weak layer is critical one. It starts earlier and propagates faster than that downslope. As soon as the tension force in the overburden layer reaches the tensile strength, the maximum upslope length of the slab is therefore given. Failure downwards, under compression, appear subsequently.

As a second question to the model of Bader and Salm, one may ask whether a linear, stress-independent viscosity can be justified for the rélative high shear stress peaks in the weak layer: The nonlinear constitutive Eqn (1) yields for pure shear

$$\eta = \left(\alpha_2 + \alpha_{22}\sigma_{21}^2 + \alpha_{222}\sigma_{21}^4\right)^{-1} \tag{34}$$

where σ_{21} is the only nonvanishing stress (approximation!). To investigate this question, an example with the following assumptions was taken up:

Shear stress in the undisturbed snowpack Eqn (31) is 4237 Pa, wherein the thickness of the overburden snow layer is 1.56 m and the slope angle 40°. The viscosity of the weak layer amounts to 10^8 Pas and that of the overburden snow 7.336 x 10^{10} Pas and thickness is 3.56 m the latter two are linear values according to Eqn (10). With the length a=5.8 m (Fig.5) in the tension part, the peak shear stress becomes 13,200 Pa. The stress-dependent viscosity is shown in Fig.6. which clearly shows that the assumption of linearity is justified, the relative decrease to the peak stress is only about 5 per cent.

4. CONCLUSION

The α values used in this paper are valid only for the tested snow (Eqn10). But it can be expected, in the sense of a hypothesis, that the mechanical characteristics are principally the same for all snow types as long as an isotropy can be assumed. For

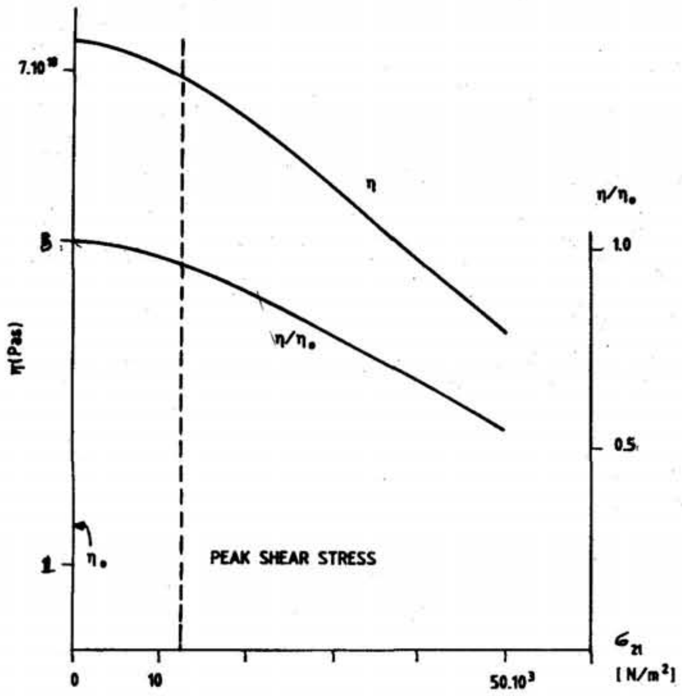

Figure 6. Stress-dependent viscosity according to Eqn (34). η_0: linear viscosity (small shear stresses. Peak shear stress according to Eqn (32).

instance the strongly nonsymmetric behaviour under compression and tension, both for axial deformation and change in volume.

To establish a three–dimensional constitutive equation tests under compression and tension are unconditionally necessary. In this context it may be remembered that on a sloping snowpack one principal stress can be tensile[3,4]. Tests on uniaxial states of stress deliver mainly dependences in relation to the first stress invariant (α values with subscripts 1). In such tests, only weak dependences exist to the second and none to the third invariant. This may be true for all snow types as stated in the above hypothesis. Therefore, triaxial tests are also needed.

Even though only a momentary behaviour of the snowstructure can be given, this is the simplest supposition one can make. For the evaluation of test results, it has to be taken into account that snow changes the viscosities considerably with only a few per cent of deformation. The snow after a test is quite different from that at the beginning. The momentary behaviour has to be backcalculated.

5. APPLICATIONS

The settlement (uniaxial state of deformation under pressure) is greatly influenced by nonlinearity (Figs 1 and 3). The same is true for tension and uniaxial states of stress (Fig.1, 2 and 4). For the mechanics of snow slab release (super-weak zone hypothesis), it could be shown that the tension zone is longer than compression zone. Consequently, the fracture propagates faster upwards than downwards. However, the nonlinearity of the shearviscosity can be neglected, which is only true if the behaviour with lower viscosities than those of Eqn (10) is principally the same (see above hypothesis).

It seems important to extend tests to different types of snow, especially towards smaller densities, say 200 - 300 kg/m³ to see whether the characteristic behaviour remains the same.

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