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SHORT COMMUNICATION

Approximate Closed-Form Solution for Projectile Trajectory and its Application to Lead Angle Computations

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ABSTRACT

In this paper, an approximate closed-form solution for trajectories of ballistic projectiles is derived. The assumption made in this derivation is to neglect the variation of the elevation angle along the trajectory in a small interval of time. The closed-form solution has been used to develop the algorithm for a lead angle computation as well as faster computation of trajectories. The fact that one of the analytical expressions, although complex, is invertible and is made use of in the algorithm.

NOMENCLATURE

C	$\rho S C_d / 2m$
ρ	Density of the air
S	Reference area of the projectile
m	Mass of the projectile
C_d	Drag coefficient
g	Acceleration due to gravity
h	Step size
s	Horizontal range of the projectile
s_t	Horizontal range of the target
V	Velocity of the projectile
V_0	Initial velocity of the projectile
u	Horizontal component of the velocity
v	Vertical component of the velocity
x_t, y_t, z_t	Target position at time t
x_d, y_d, z_d	Target present position
u_d, v_d, w_d	Target present velocity
z	Height of the projectile
γ	Elevation angle
γ_0	Initial elevation angle
ψ_0	Lead azimuth angle
t	Time

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1. INTRODUCTION

Computation of lead angles and preparation of range tables for ballistic projectiles are some of the essential tasks in many theoretical and practical applications, such as lead computing sights, vulnerability study of aircraft, etc. In these tasks, trajectories of the projectiles like bullets, shells and missiles are to be computed several times. These computations consume substantial amount of computational time. The aim of the present study is to reduce the lead angle computational time to less than allotted time of 0.1 s.

Governing equations of these trajectories are a system of coupled ordinary differential equations and are solved only numerically. Some attempts were made in the past to simplify these equations and find analytical solutions. One such attempt was that of Siacci[1], who made two assumptions: (i) approximation of ratio of cosines of initial elevation and elevation of any point on the trajectory to unity, and (ii) neglecting variations in the air density along the trajectory. With these assumptions, the governing equations are reduced to a simpler form and can be integrated by means of quadratures. Application of this theory is limited to trajectories with relatively small elevation angles.

In Siacci theory, the drag function is taken as proportional to n^{th} power of the velocity, where values of n are given in a Mayevski's table [1]. The values of n vary from 1.55 to 5.00 depending upon the velocity of the projectile. In many applications, the drag is also taken to be proportional to the square of the velocity. The proportionality constant includes drag coefficient. In this paper, an approximate analytical solution for the trajectories has been derived by taking the latter form of drag function and making an approximation similar to the first approximation of Siacci theory. Taking variation of the elevation angle to be negligible in a small interval of time, the governing equations are decoupled. The resulting equations are then integrated to yield a closed-form solution.

Trajectories of the projectiles constructed using this approximate solution were analysed. Error analysis showing the truncation error had been conducted. Numerical experiments were conducted to compare such computed trajectories with those computed through numerical integration using Runge-Kutta method. As the step size decreases, the trajectories of the new method converge to those of the numerical method. The new method is faster than the numerical method.

This paper also contains a new faster algorithm based on the closed-form solution for determining the firing angles to intercept a moving target. This algorithm can be used in the lead computing sights. The closed-form solution for the projectile trajectories, derived in Section 2 and Section 3, outlines a method of constructing trajectories using the closed-form solution. It also includes details of the numerical results and the error analysis. In Section 4, an algorithm for lead angle computation that uses the closed-form solution is described.

2. CLOSED-FORM SOLUTION OF PROJECTILE TRAJECTORY

Trajectories of the projectile are the two-dimensional curves with the following governing equations:

$$\frac{du}{dt} = -CV^2 \cos(\gamma) \tag{1}$$

$$\frac{dv}{dt} = -CV^2 \sin(\gamma) - g \tag{2}$$

$$\frac{ds}{dt} = u \tag{3}$$

$$\frac{dz}{dt} = v \tag{4}$$

It is assumed that the variation in the elevation angle in a small time interval, $(t, t+h)$ is negligible. Without loss of generality, a typical time step to be the initial time step, that is $(0, h)$ is taken. This assumption reduces Eqns (1) and (2) to

$$\frac{du}{dt} = - C \sec(\gamma_0)\, u^2 \tag{5}$$

$$\frac{dv}{dt} = - C \operatorname{cosec}(\gamma_0)\, v^2 - g \tag{6}$$

where γ_0 is the elevation angle at $t = 0$.

These equations are now decoupled and can be integrated to get a closed-form solution. Depending upon the signs of right hand side constant coefficients of Eqn (6), two different cases arise in the integration. The first case is of the ascending mode of flight, i.e., when the elevation angle is positive and the coefficients have same sign. The other case is of the descending mode of flight, i.e., when the elevation angle is negative and the coefficients have different signs.

By taking γ_0 to be positive and the initial condition as $t=0$, $V=V_0$, $u=V_0 \cos(\gamma_0)$ and $v=V_0 \sin(\gamma_0)$, integration of the Eqns (5), (6) and (3), (4) yield

$$u = \frac{V_0 \cos(\gamma_0)}{CV_0 t + 1} \tag{7}$$

$$v = \sqrt{\frac{g \sin(\gamma_0)}{C}} \tan\left(\sqrt{\frac{Cg}{\sin(\gamma_0)}}\,(c_1 - t)\right) \tag{8}$$

$$s = s_0 + \frac{\cos(\gamma_0)}{C} \log(1 + CV_0 t) \tag{9}$$

$$z = z_0 + \frac{\sin(\gamma_0)}{C} \log\left\{\cos\left(\sqrt{\frac{Cg}{\sin(\gamma_0)}}\,(c_1 - t)\right) \Big/ \cos\left(c_1 \sqrt{\frac{Cg}{\sin(\gamma_0)}}\right)\right\} \tag{10}$$

where (s_0, z_0) is a point on the trajectory at $t = 0$ and

$$c_1 = \sqrt{\frac{\sin(\gamma_0)}{Cg}} \tan^{-1}\left(V_0 \sqrt{\frac{C \sin(\gamma_0)}{g}}\right) \tag{11}$$

In the descending mode of flight, γ_0 is negative and hence the coefficients of Eqn (6) differ in signs.

As far as Eqn (5) is concerned, there is no change between the ascending and the descending modes. Therefore, the expressions for u and s in the descending mode are same as those in the ascending mode.

By taking γ_0 to be negative, and integrating Eqns (6) and (4), one gets

$$v = -\sqrt{\frac{-g \sin(\gamma_0)}{C}} \tan h\left(\sqrt{\frac{-Cg}{\sin(\gamma_0)}}\,(c_2 - t)\right)$$

$$z = z_0 + \frac{\sin(\gamma_0)}{C} \log \left[\cos h\left(\sqrt{\frac{-Cg}{\sin(\gamma_0)}}\,(c_2 - t)\right) \Big/ \cos h\left(c_2 \sqrt{\frac{-Cg}{\sin(\gamma_0)}}\right)\right]$$

where

$$c_2 = \sqrt{\frac{\sin(\gamma_0)}{-Cg}} \tan h^{-1}\left(V_0 \sqrt{\frac{-C \sin(\gamma_0)}{g}}\right) \tag{14}$$

3. TRAJECTORY CONSTRUCTION USING ANALYTICAL SOLUTION

3.1 Single-Step Method

Generally the trajectories are constructed using a suitable time step. The positions of the projectile after every time step are calculated, and the curve joining these points gives the trajectory. The above derived analytical expressions can be used in the calculation of the trajectory points as follows:

For a given initial elevation angle (γ_0) and velocity (V_0), the trajectory point after time (h) is computed using Eqns (7) to (11) or Eqns (7) to (9), and (12) to (14) depending upon whether γ_0 is positive or negative, respectively. While calculating a trajectory point after next time step, elevation angle and velocity of the previous step is taken as γ_0 and V_0, respectively.

If values of the drag coefficient and the air density versus velocity are given in a tabular form then values of these coefficients in each time interval can be computed by an interpolation method. In this paper, the trajectories constructed by the above method are referred as constructed trajectories.

3.2 Numerical Experiment & Error Analysis

The above method of constructing trajectories is a single-step method. Order of this method is estimated for a numerical example.

Let $s_0(t)$ and $z_0(t)$ be the range and height of a point on an actual trajectory of the projectile for a particular initial elevation. Let $s_h(t)$ and $z_h(t)$ be the same quantities of a corresponding point on the constructed trajectory of the projectile. Then cumulative errors $E_s(t)$ and $E_z(t)$ in $S_h(t)$ and $z_h(t)$, respectively, in the power of h are expressed as

$$E_s(t) \equiv s_h(t) - s_o(t) = d_1 h + d_2 h^2 + d_3 h^3 + \dots, \tag{15}$$

$$E_z(t) \equiv z_h(t) - z_0(t) = e_1 h + e_2 h^2 + e_3 h^3 +$$

where d and e are constants. The largest integer p such that

$$h^{-1} E | = O(h^p)$$

then the order of the method[2] is p.

The error terms for $p = 1, 2, 3$ and 4 have been estimated. For $p = 1, 2, 3$ it has been observed that the p^{th} term dominates, while remaining terms are almost negligible or sum of them is negligible. The magnitudes of the error terms versus time for $p = 3$ in a typical case[3], are plotted in Figs 1 and 2. Here, the continuous lines represent the first-order term,

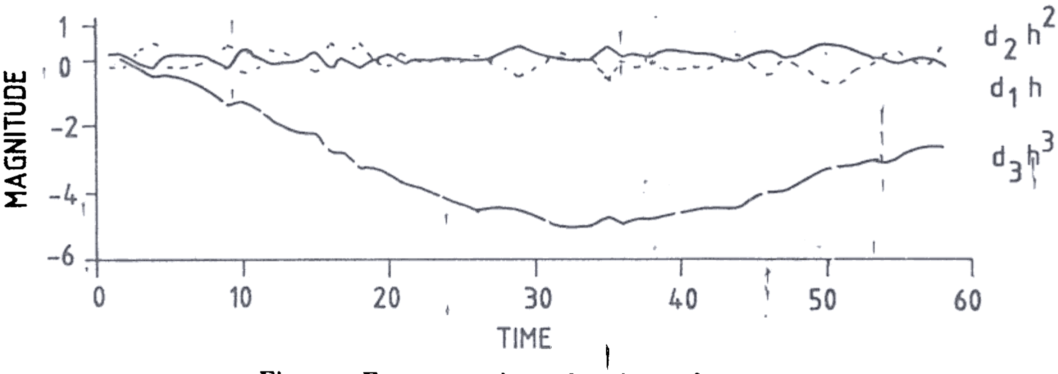

Figure Error terms in s values ($\gamma_0 = 45°$ and $h_0 = 0.125$)

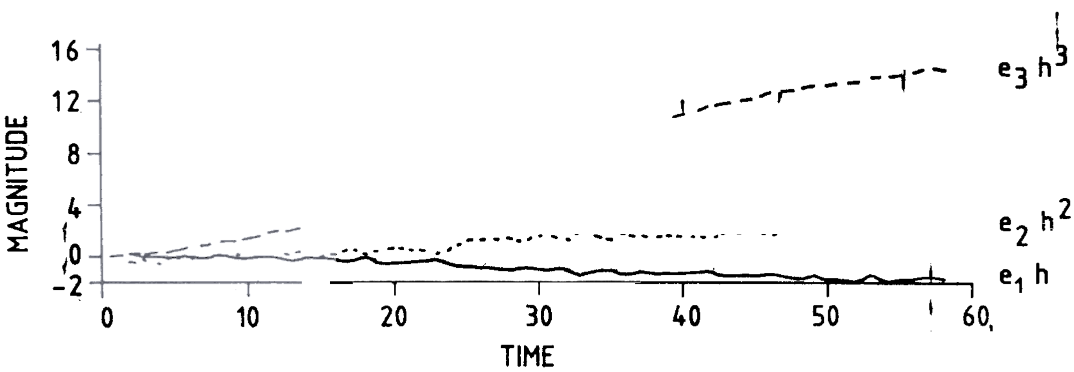

Figure 2. Error terms in z values ($\gamma_0 = 45°$ and $h_0 = 0.125$)

dotted lines the second-order term and dashed lines the third-order term. The continuous and dotted lines are almost reflection of one another wrt zero-line. Overall effect due to these two terms is negligible. The third-order term dominates. This shows that the truncation error is of the third-order. Hence, the method is a second-order method.

Further, the constructed trajectories have been compared with a reference trajectory computed applying the Runge-Kutta method using the following measure of deviation:

$$Dev. = \sqrt{\frac{(s - \bar{s})(s - \bar{s}) + (z - \bar{z})(z - \bar{z})}{\bar{s}\bar{s} + \bar{z}\bar{z}}} \qquad (18)$$

The barred quantities are corresponding to the reference trajectory. In Fig. 3, maximum values of the deviation of analytical trajectories in the entire flight time are plotted versus step size. As the step size decreases, the maximum deviation decreases showing convergence of constructed trajectories to the reference trajectories. It was also found that the computational time of constructed trajectories is about 15 per cent less than those obtained by the Runge-Kutta fourth-order method with the same step size.

4. ALGORITHM FOR LEAD ANGLE COMPUTATION

A new faster algorithm for lead angle computation based on the closed-form solution has been described. This algorithm takes current position and tracking speed of the target as inputs. Assuming straight-line path for the target, it computes required firing angles for a possible interception with a projectile to be launched from the origin.

4.1 Steps of the Algorithm

The algorithm consists of two phases of computations. In the first phase, an approximate time for the target and projectile to arrive at an equal horizontal range is determined. In the second phase, a projectile trajectory is found, such that the miss-distance is within the given limit. First six steps given below are of the phase 1 of the algorithm. The remaining steps are of phase 2.

Step The given present point of the target be (x_d, y_d, z_d). From the tracking speed, velocity (u_d, v_d, w_d), of the target is computed.

Step 2 Using a suitable time step, say dt_1, the time (t) required for the target and the projectile to arrive at equal horizontal range is determined. Initially, t is taken to be equal to dt_1.

Step 3 Position of the target at t is computed as

$$\begin{aligned} x_t &= x_d + u_d t \\ y_t &= y_d + v_d t \qquad (19) \\ z_t &= z_d + w_d t \\ s_t &= \sqrt{x_t^2 + y_t^2} \end{aligned}$$

where s_t is horizontal range

Step 4 The γ_0 of the gun is taken as

$$\gamma_0 = \tan^{-1}\left(\frac{z_t}{s_t}\right)$$

Step 5 The projectile position (s, z) at t is computed from Eqns (9) and (10). In these computations, the whole time of flight of the projectile, i.e., t is taken as a single-time interval.

Step 6 If t is equal to dt_1, then add dt_1 to t and go back to *Step* 3. If the distance between the target and the projectile is not decreasing as time increases then an interception may not be possible, and stop. Otherwise, if s is greater than s_t, go to next step, else add dt_1 to t and go back to *Step* 3.

Step 7 Increment γ_0 by

$$d\gamma = \tan^{-1}\left(\frac{z_t - z}{s_t}\right) \qquad (20)$$

Step 8 By dividing t into small intervals of size dt_2, the projectile trajectory is computed till its range becomes close to s_t but less. Let s and z be the horizontal range and height of such a position, respectively. Let t_c, V_c and γ_c be the time of flight, current velocity and elevation of the projectile trajectory, respectively.

Step 9 If γ_c is non-positive, then an interception may not be possible in the ascending mode of projectile flight, and stop. Otherwise go to next step.

Step 10 Let $ds = s_t - s$. Time dt required for the projectile to traverse further the horizontal range equal to ds, is computed using inverted form of Eqn (9), i.e.,

$$dt = \frac{1}{CV_c}\left(\exp\left(\frac{Cds}{\cos(\gamma_c)}\right) - 1\right)$$

Step 11 If $(t_c + dt)$ is very close to t, then go to next step. Otherwise set $t = t_c + dt$ and recalculate s_t for time t using Eqn (19) and go back to the previous step.

Step 12 The height of the target (z_t) in t is computed using Eqn (19). The incremental height dz achieved by the projectile in time dt is computed using Eqn (10). If difference between z_t and $(z + dz)$ is greater than given miss-distance then go back to *Step* 7. Otherwise, the current position of the target may be taken as a possible impact point. The current value of γ_0 is the required lead elevation angle of the gun. The azimuth angle of the impact point gives required lead azimuth angle.

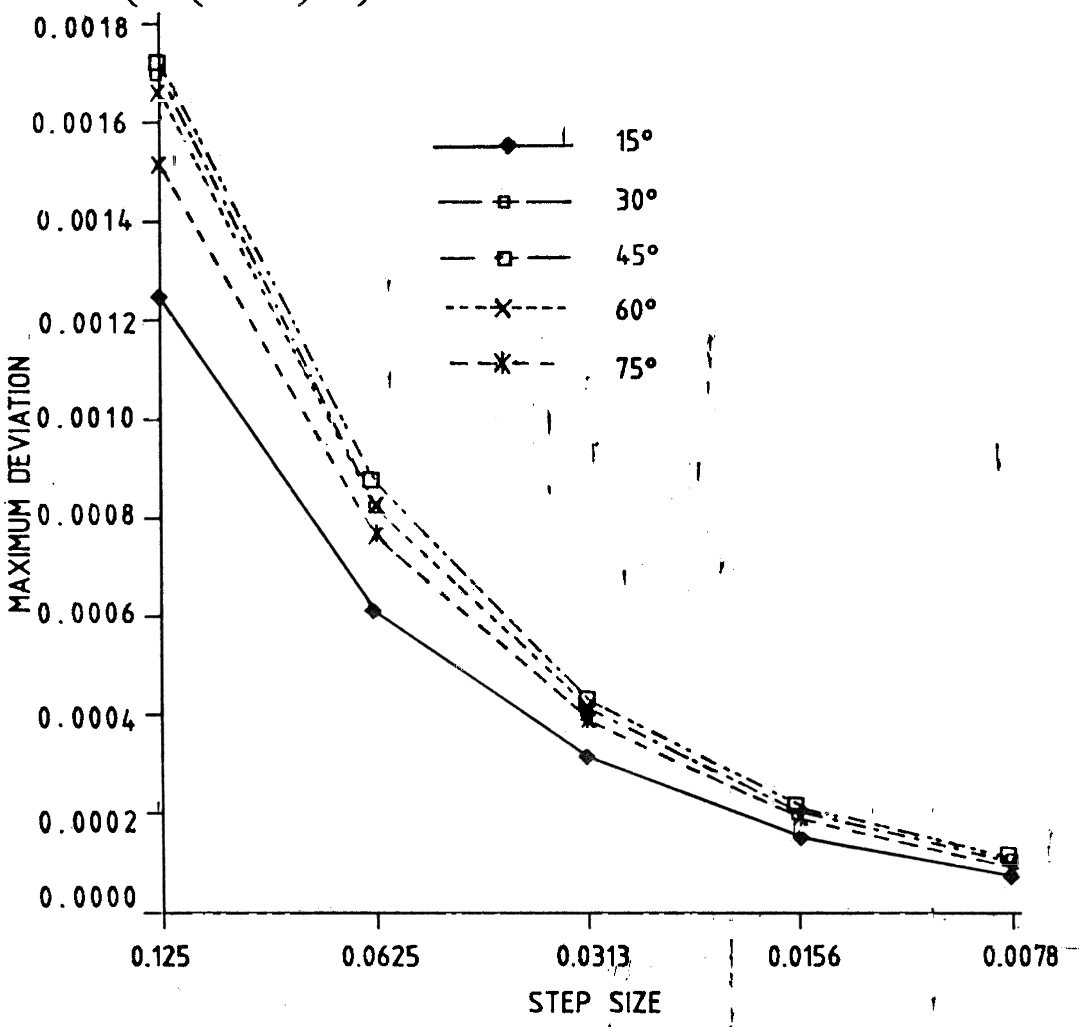

Figure 3. Maximum deviation versus step size plots of trajectories with various initial elevation

The increment given to γ_0 in *Step* 7 is based on a well-known fact in ballistics. Let d be the vertical miss-distance between a projectile trajectory and a point above the trajectory. Let β be the initial elevation of the trajectory and α, the angle subtended by d at the origin. A trajectory of the projectile with an initial elevation angle of $\alpha + \beta$, is closer to the point than the previous trajectory.

4.2 Results

An experiment has been conducted to evaluate the performance of the algorithm considering a typical shell. An arbitrary point in space is taken as the present position of a target, and an arbitrary target tracking speed is assumed. A sample set of such target positions and tracking speeds in spherical coordinate system is given in Table 1.

Table 1. Input data in spherical coordinate system

Data No.	Current target position			Rate of change of		
	Range (m)	Azimuth (deg)	Elevation (deg)	Range (m)	Azimuth (deg)	Elevation (deg)
	3000	0	5	−350	−4	2.5
2	3400	60	8	−350	−3	2.0
	3800	120	11	−350	−2	1.0
4	4200	180	14	−350	−1	0.5
5	4600	240	17	−350	0	0.0
6	5000	300	20	−350		0.5
	5400	360	23	−350	2	1.0
8	5800	420	26	−350	3	−2.0
9	4500	480	29	−350	4	−2.5
10	4500	540	32	−350	0	−4.0

Table 2 contains lead angles computed using the algorithm for sample data points of Table 1. In these computations, $dt_1 = 0.1$, $dt_2 = 0.05$ and miss-distance $d = 0.01$ m have been used. The miss-distances and times of flight are calculated offline, once the lead angles are computed by the algorithm. In the offline calculations, the Runge-Kutta fourth-order numerical integration scheme is used to compute the shell trajectory. This is to validate the algorithm. Although actual miss-distances are high compared to computed miss-distance, they are acceptable for the order of the ranges considered.

Table 2. Performance results of the algorithm

Data No.	Firing angles		Time of flight (s)	Miss-distance (m)	Computation time (s)
	Azimuth (deg)	Elevation (deg)			
1	−13.28	13.77	2.61	1.61	0.0077
2	48.13	16.45	3.02	0.77	0.0110
3	110.88	16.37	3.42	0.75	0.0126
4	174.79	17.65	3.84	3.31	0.0137
5	240.00	18.27	4.28	3.12	0.0187
6	−53.22	24.59	4.78	1.52	0.0214
7	14.45	16.60	5.37	0.54	0.0231
8	84.02	8.91	6.31	3.50	0.0264
9	141.72	14.12	4.47	2.47	0.0220
10	180.00	9.70	4.41	2.03	0.0187

5. CONCLUSION

The results validate the closed-form solution. The fact that the solution is invertible may increase its usefulness in many applications.

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REFERENCES

Bliss, G.A. Mathematics for exterior ballistics. John Wiley and Sons, London, 1944.

2. Jain, M.K.; Iyengar, S.R.K. & Jain, R.K. Numerical methods. Wiley Eastern Limited. 1994.

3. Singh, V.P. & Ganagi, M.S. Approximate closed-form solution for two-dimensional trajectories of projectiles. Centre for Aeronautical Systems Studies & Analyses, Bangalore, Report No. CASSA-S-97-4.