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THE COMPARATIVE EFFICIENCY OF THE X & R CHARTS AND L-S CHART FOR STATISTICAL QUALITY CONTROL

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ABSTRACT

This paper deals with the comparative efficiencies of the $\bar{X}$ and R charts and the L-S chart (L being the largest and S the smallest value of the sample) from the theoretical and practical points of view. It has been found that in practice the corrective steps suggested by both tallied almost exactly. From theoretical points, when standard deviation varies, the L-S chart is almost as efficient as the X and R charts, but with the variation in the mean the relative efficiency of the L-S chart comes down appreciably as the mean shifts more and more from the original value.

Introduction

The introduction of SQC as a routine measure in the factory at important stages of production enables the producer to take corrective steps in time so as to reduce the percentage of defectives coming out of the machines. It is one of the important ways of reducing the cost of production. It also provides the management with information regarding the efficiencies of different workers, machines and methods of operation.

At present the SQC is done by the well-known $\bar{X}$ and R charts for measurable characteristics, the p-chart for qualitative characteristics and the c-chart for the number of defects per unit of products. The $\bar{X}$ and R charts are the most powerful of the three SQC tools. They are used extensively in many factories for controlling production. It has, however, been found that some time elapses before these charts are ready for inspection and this results in some delay in taking corrective steps and this delay naturally increases the number of defectives produced by a machine. Besides this, the machine operator who is more accustomed with the individual measurement and its tolerance than with the sample mean and range, may not be able to interpret these charts properly at the required time. In view of these disadvantages, it is desirable to examine other methods of control which are simpler and for all practical purposes as efficient as the conventional $\bar{X}$ and R charts. Howell (1949) has investigated the merits of the X and R charts with the L-S chart, where L and S represent the largest and the smallest value of the sample, for SQC work. He finds that the L-S chart is almost as efficient as the $\bar{X}$ and R charts for all practical purposes. Stevens (1946) after some elaborate investigations concludes that the control of the mean and standard deviation can be done by control charts of $c-a$ and $c+a$ respectively, where a is the number of articles passing the smaller gauge and c is the number failing to pass the larger gauge. The sizes of the gauges are so fixed that under normal production conditions the upper gauge will not allow a fixed percentage of articles to pass through it while the lower gauge will allow another fixed percentage of the articles to pass through it. This method is as efficient as the control by the $\bar{X}$ and R charts provided ten articles are gauged in place of eight exactly measured.

For want of sufficient supply of gauges, control by gauging does not appear to be feasible in Indian factories at present. The L-S chart, however, appears to be well suited for adoption in the Indian factories. But before we recommend this method it is essential that a critical examination of the two methods should be made. The purpose of the present paper is to investigate the relative efficiencies of the two SQC methods, namely, $\bar{X}$ and R charts and the L-S chart, both from experimental and theoretical points of view.

THEORY OF $\bar{X}$ AND R CHARTS AND L-S CHART

(a) $\bar{X}$ and R charts

The $\bar{X}$ and R charts have been adopted as the tools for SQC work on the assumption that the samples taken at different intervals come from a normal population. The normal distribution involves two parameters m(mean) and σ(standard deviation—S.D.). Our object in SQC is to control these two parameters. Now the sample mean $\bar{X}$ is distributed normally with mean m and standard deviation $\sigma/\sqrt{n}$ where n is the size of the sample. For this distribution the central confidence interval at the probability level ·998 is $m \pm 3{\cdot}09\, \sigma/\sqrt{n}$ *i.e.*, the mean of 998 out of 1,000 samples are expected to be within these limits. Taking $\bar{\bar{X}}$ and $\bar{R}/d_2$ as the estimates for m and σ respectively (where $\bar{\bar{X}}$ is the average sample mean, $\bar{R}$ the average range calculated from an initial set of samples and d_2 the expected ratio of range to σ), the above limits reduce to $\bar{\bar{X}} \pm 3{\cdot}09 \dfrac{\bar{R}}{d_2\sqrt{n}}$ or $\bar{\bar{X}} \pm A_2 \bar{R}$, where $A_2 = \dfrac{3{\cdot}09}{d_2\sqrt{n}}$. These two limits are called the upper control limit (U. C. L.) and lower control limit (L. C. L.) for the $\bar{X}$ chart. For detecting any changes in the dispersion the range chart is used. The upper control limit for the range chart is $D\bar{R}$ where $D = \omega/d_2$ (ω being equal to R/σ). The factors A_2 and D have been calculated on the basis of the tables for ω and d_2 prepared by Tippett (1925) and Pearson (1942). Lack of control will be indicated when the sample mean or the range (or both) fall outside the control limits.

The modified control limits for the mean chart

When the machine is too fine for the job (*i.e.*, if the process variation is much smaller than the tolerance) we shall use modified control limits because the use of the ordinary control limits is not economical.

The modified control limits for the mean chart are

$$\text{U. C. L.} = T_U - K\bar{R}$$
$$\text{L.C.L.} = T_L + K\bar{R}$$

where T_U = upper tolerance limit.
T_L = lower tolerance limit.

$$\text{and } K = \left(3{\cdot}09 \frac{1}{d_2} - 1{\cdot}96 \frac{1}{d_2\sqrt{n}}\right)$$

The modified control limits will be much wider than the ordinary control limits. The table givingthe values of A_2, D and K for different sample sizes are given by Sealy (1945), Quality Control for Engineers.

The question naturally arises as to whether the $\bar{X}$ and R charts described above can be used when the parent population is not normal. So far as the $\bar{X}$ chart is concerned this can be applied even if the parent population is not normal because in the majority of populations the sample mean is distributed normally provided the sample size is not small. As regards the R chart, for all practical purposes it is used without any modification.

(b) L-S chart

As in the case of $\bar{X}$ and R charts, the theory of L-S chart also depends on the assumption that the parent population is distributed normally.

Let L and S be the largest and smallest values respectively of a sample of size n. Then for a symmetrical distribution.

$$E(L+S)/2=m,$$

where m is the mean of the population. Moreover we know that

$$E[(L\text{-}S)=R]=d_2\sigma \text{ (Tippet 1925)}$$

Hence
$$E(L)=m+d_2\sigma/2$$
$$E(S)=m-d_2\sigma/2$$

The standard deviation of the extreme values has also been calculated by Tippet (1925). These are

$$\sigma_{\bar{L}}=\sigma_{\bar{S}}=d_4\sigma$$

Hence the upper and the lower control limits for the largest and smallest value respectively are

$$E(L)+3\sigma_{\bar{L}}=m+(\frac{d_2}{2}+3d_4)\sigma$$
$$=m+A_4\sigma$$

$$\& \quad E(s)-3\sigma_{\bar{s}}=m-(\frac{d_2}{2}+3d_4)\sigma$$
$$=m-A_4\sigma$$

where
$$A_4=(\frac{d_2}{2}+3d_4)$$

putting $(\bar{L}+\bar{S})/2=M$ and $\bar{R}/d_2$ as the estimates for m and σ respectively, the estimated control limits for the L-S chart reduce to .

$$U.C.L.=M+A_3\bar{R}$$
$$\& \quad L.C.L.=M-A_3\bar{R}$$

where $A_3=(0\cdot5+3\frac{d_4}{d_2})$

The values of A_3 for different sample sizes have been given by Howell (1949) This has been reproduced in our lecture notes on " Statistical Quality Control and Sampling Inspection " (1953).

EXPERIMENTAL INVESTIGATIONS

The experiment consisted in trying to control a process which was entirely out of control by the help of the L-S chart and $\bar{X}$ and R charts and comparing the efficiencies on the basis of the corrective steps suggested by both the methods.

(a) Selection of component and characteristic

The job selected for study was the component of a fuze manufactured by a single spindle automatic machine at the Gun and Shell Factory, Cossipore from coiled brass rods of diameter ·1875 inches. The characteristic studied was the total length of the component. The reasons for selecting this characteristic were ease of measurement, high tolerence and technical importance. When the length of the component is low, the fuze is likely to function prematurely and thus prove dangerous to the users and hence it is a critical defect. On the other hand when the length is high, the fuze will not function and thus it is a major defect.

(b) A short description of the manufacturing process

There are two spindles, one placed inside the other. On the face of the outer spindle there is a collet. A feed finger is attached to the face of the inner spindle. The other end of the inner spindle is attached to the feeding crank. The brass rod of diameter ·1875″ is pushed into the inner spindle. It passes through the feed finger and comes up to the face of the collet. With the forward stroke of the crank the feed finger enters into the collet along with the rod and as a result a certain length of the rod is pushed out. The feed finger now comes back to its previous position but the rod cannot come back as it is gripped by the collet. At the same time a stopper falls in front of the collet and adjusts the length. After a few more operations a tool parts off the job from the rod. Hence the length of the component depends on the following factors :

(*i*) setting of the stopper,

(*ii*) setting of the parting tool,

(*iii*) condition of the feed finger and the collet,

(*iv*) setting of the feeding crank.

Low length of the component is due to short feeding and may occur in the following cases :

(*i*) when the setting of the crank is low,

(*ii*) when the feed finger is loose and results in the slipping of the rod,

(*iii*) when the collet is slack in which case the rod may come back along with the backward movement of the crank.

But high length is only due to high setting of the stopper.

(c) Method of collection of data

The investigation lasted for twenty three days from 4th August, 1953 to 26th August, 1953. Samples of consecutive five jobs were taken at a regular interval of half an hour. Each component was measured carefully with a micrometer after proper cleaning and removing of bars, if any. The measurements were recorded on a data sheet. Any adjustments of the machine were noted in the remarks column of the sheet. For each sample the largest and smallest values were plotted on a graph against the tolerance limits given in the specification. The control limits were calculated and shown on the chart given in Fig. 1.

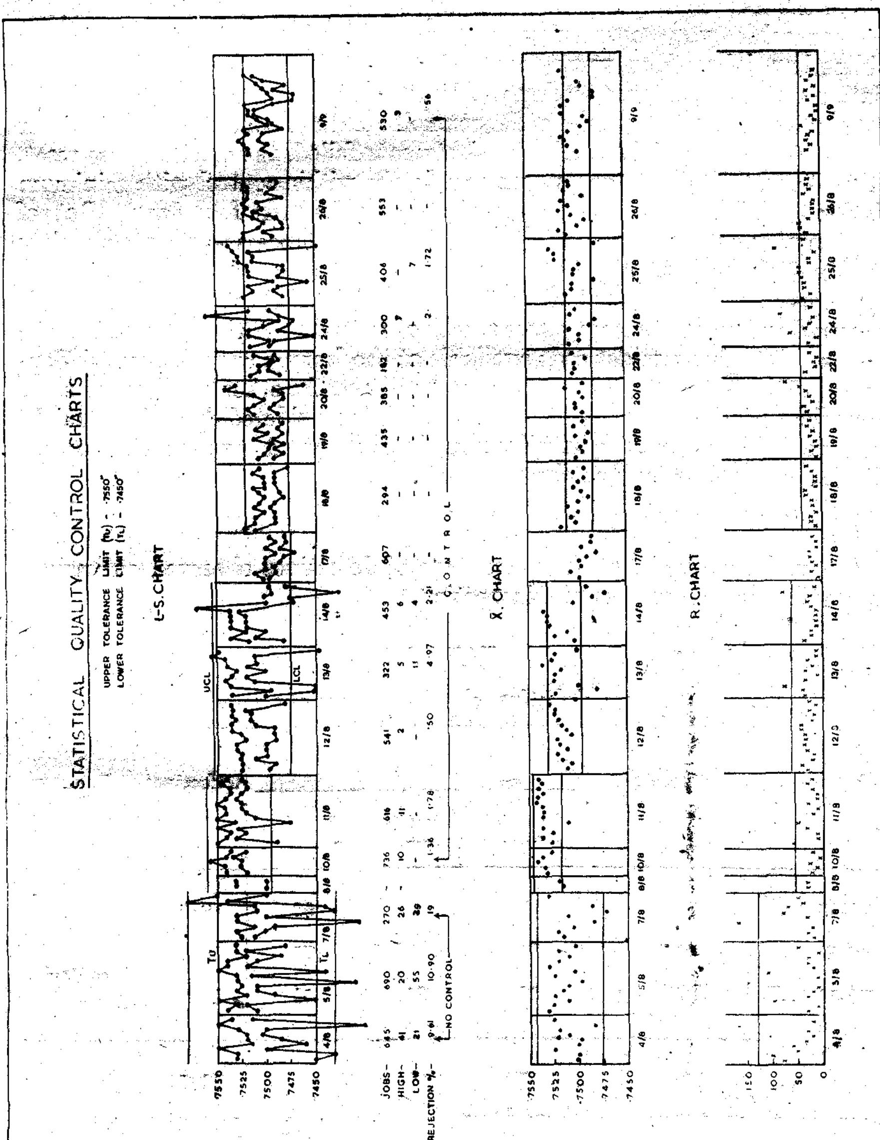

Fig. 1.

Similarly the mean and range for each sample were calculated and plotted on the respective control chart. The new pair of control limits were calculated as the quality improved. The corrective steps suggested by the L-S chart were followed. As a result of control the percentage of defectives went down appreciably.

(d) A brief description of the control chart

For the first three days the process was completely out of control, the variation was very high in comparison to the tolerance limits, the two control limits of the L-S chart were outside the specification limits. On 10th August, 1953 the machine was properly examined and set on a higher side with a view to avoiding low jobs. The work improved to a great extent. Because of low variation in the process the setting was lowered on 12th August, 1953 and the process was completely brought under control. Lack of control was noticed for the next two days. When minor adjustments failed, the machine was reset. On 17th August, 1953 the process was completely within control with minimum variation and the state of control continued for the next four days. Out of control points appeared on 24th August, 1953 but the process was beaten back to control on 26th August, 1953. For all these days the coiled rods after straightening were used, and in the opinion of the technicians these undulated rods were considered to be the main cause of lack of control. With a view to investigate this point a few straight rods available at the factory were fed on 9th September, 1953. The work did not improve at all ; on the other hand due to some reasons the variation increased slightly. But it is considered that with straight rods it would be easier to maintain control and repeated examination of the feeding arrangements may not be necessary.

(e) The percentage of defectives during the period of investigation

The table below shows the percentage of defectives observed on various dates on the basis of cent per cent inspection.

TABLE I

Percentage defectives observed on various dates

Date	Jobs inspected	Defectives			% Defectives	Remarks
		Length high	Length low	Total		
4-8-53 ..	645	41	21	62	9·61	No control
5-8-53 ..	690	20	55	75	10·90	
7-8-53 ..	270	26	25	51	19·00	
Total ..	1605	87	101	188	11·71	

TABLE I—*contd.*

Date	Jobs inspected	Defectives			% Defectives	Remarks
		Length high	Length low	Total		
10-8-53 ..	736	10	..	10	1·36	Control
11-8-53 ..	616	11	..	11	1·78	
12-8-53 ..	541	2	..	2	·50	
13-8-53 ..	322	5	11	16	4·97	
14-8-53 ..	453	6	4	10	2·21	
17-8-53 ..	607	..	..	..	..	
18-8-53 ..	294	..	..	..	..	
19-8-53 ..	436	..	..	..	..	
20-8-53 ..	385	..	..	..	..	
22-8-53 ..	182	..	..	..	..	
24-8-53 ..	300	7	..	7	2·33	
25-8-53 ..	406	..	7	7	1·72	
26-8-53 ..	533	..	..	..	..	
9-9-53 ..	530	3	..	3	·56	
Total ..	6361	44	22	66	1·04	

The above table shows that the percentage of defectives in the beginning was on the average 11·71 while due to control it was brought down to 1·04.

(f) Comparison between the L-S chart and the $\overline{X}$ & R charts

(*i*) *L-S chart detect lack of control almost equally efficiently as the $\overline{X}$ & R charts.* Throughout this investigation the process was controlled by taking corrective steps as suggested by the L-S chart. This was checked with the $\overline{X}$ & R charts. The results of the interpretation of both the charts were substantially the same.

(*ii*) *In L-S chart a simple comparison of the control limits with the tolerance limits ensures economic control while in $\overline{X}$ & R charts such assurance requires further calculation and comparison.* If the two control limits of the L-S chart are within the tolerance limits, so long as the process is within control we are fairly sure that the job will meet the specifications. This is not so obvious in the case of $\overline{X}$ & R charts.

(*iii*) *In some cases quicker diagnosis of the assignable cause may be possible with the L-S chart.* The last sample of 20th August, 1953 (*vide* S.Q.C. Charts) indicates lack of control on both sides of the control limits. This suggested the adjustment of the position of the stopper and the feeding arrangements. But according to $\bar{X}$ & R charts the range only was out of control and thus indicated that the variation had increased. This might have been either due to irregular feeding or sudden shift of the stopper. Similar situations were noticed with the last samples of 13th August, 1953 and 25th August, 1953. In such cases the L-S chart suggests definitely the nature of corrective steps while the $\bar{X}$ & R charts keep us in doubt.

(*iv*) *L-S chart is much simpler.* In the case of $\bar{X}$ & R charts the mean and range are to be calculated and plotted. In L-S chart, only the largest and smallest values are to be noted and plotted. Hence the L-S chart is much simpler than the $\bar{X}$ & R charts and will take much less time to maintain.

(*v*) *L-S chart is readily explainable to the machine operator.* The machine operator is more accustomed with individual measurement and the tolerance on it than with the sample mean and range which are confusing to him and therefore he can more easily understand the L-S chart than the $\bar{X}$ and R charts.

Theoretical Investigation

Some idea of the relative efficiencies of the SQC methods considered above can be obtained by comparing the probabilities of the samples drawn at different times, with varying m and σ lying within the respective control limits established for given m and σ.

For given m and σ let C_1, C_2 and R be control limits for the mean and the range charts for the probability levels P_1 and P_2 respectively. Then $P_1 \times P_2$ is the probability that the sample will lie within the control limits both in the mean and the range charts.

Let L_1 and S_1 be the upper and lower control limits for the *L-S* chart for the same m and σ at the probability level P_3 such that $P_1 \times P_2 = P_3$. Assuming $P_1 = \cdot 998$, $P_2 = \cdot 998$, $P_3 = P_1 \times P_2 = \cdot 996$, $m = \cdot 7500''$, $\sigma = \cdot 0010''$, C_1, C_2, R, L_1 & S_1 are given below for the characteristic investigated for $n = 4$ and 5.

	$n = 4$	$n = 5$
C_1	·7515″	·7514″
C_2	·7485″	·7486″
R	·00478	·00523
L_1	·7533″	·7534″
S_1	·7467″	·7466″

These values are obtained by solving the following equations.

$$
\left.
\begin{aligned}
P_1 &= \int_{\frac{C_1-m}{\sigma/\sqrt{n}}}^{\frac{C_2-m}{\sigma/\sqrt{n}}} \phi(t)\,dt \;; \qquad P_2 = \int_0^R \psi(R)\,dR \\
P_3 &= \left[\int_{\frac{S_1-m}{\sigma}}^{\frac{L_1-m}{\sigma}} \phi(t)\,dt \right]^n \;; \qquad \phi(t) = \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}
\end{aligned}
\right\} \quad (A)
$$

The probability integrals for the range viz, P_2, are discussed by Pearson (1942).

The probabilities of accepting samples with varying m and σ for the control limits given above for the $\overline{X}$ and R charts and the L-S chart are given in Table II for n=4 and n=5.

TABLE II

Probability of accepting samples with $\overline{X}$ and R charts and L-S chart with usual control limits.

n	m	σ	P_1	P_2	P_1P_2	P_3
4	·7500	·0010	·9980	·9980	·996	·996
	,,	·0015	·9545	·8910	·851	·889
	,,	·0020	·8664	·6710	·581	·670
	,,	·0025	·7699	·4694	·361	·437
	,,	·0030	·6827	·3255	·222	·282
	·7505	·0010	·9772	·9980	·975	·990
	·7510	·0010	·8186	·9980	·817	·960
	·7515	·0010	·4091	·9980	·408	·864
	·7520	·0010	·0869	·9980	·087	·656
	·7525	·0010	·0062	·9980	·006	·364
5	·7500	·0010	·9980	·9980	·996	·996
	,,	·0015	·9634	·9020	·869	·889
	,,	·0020	·8969	·6566	·589	·627
	,,	·0025	·7923	·4231	·335	·385
	,,	·0030	·7063	·2664	·188	·227
	·7505	·0010	·9801	·9980	·978	·990
	·7510	·0010	·8186	·9980	·817	·960
	·7515	·0010	·4091	·9980	·408	·864
	·7520	·0010	·0869	·9980	·087	·656
	·7525	·0010	·0062	·9980	·006	·364

In the above table P_1, P_2 and P_3 are the probabilities of accepting samples by the $\overline{X}$ and R charts and L-S chart respectively for different values of m & σ and for the control limits obtained in (A).

The curve obtained by plotting the probabilities for m or σ is called the operating characteristic (o.c) curve and is shown in Fig. 2.

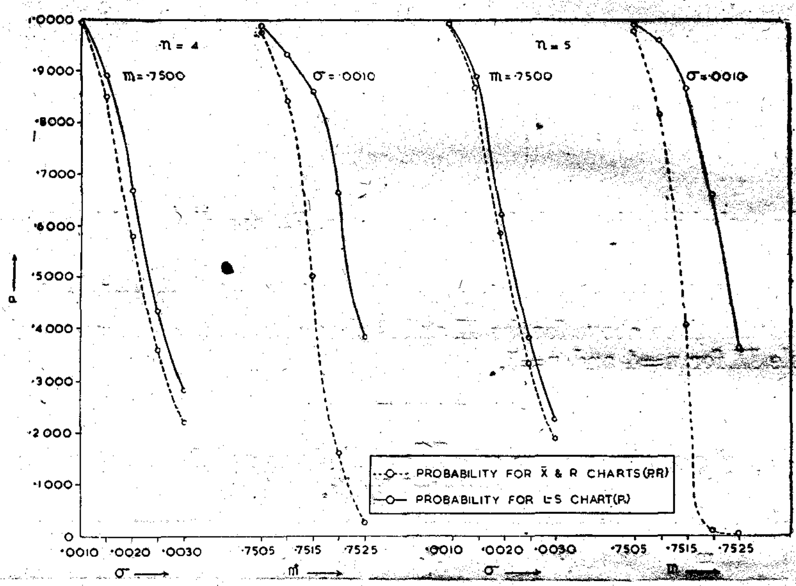

FIG 2

In actual practice it is more economical to use the modified control limits for the mean chart when the variation in production is very small and therefore it would be more useful if the relative efficiencies are compared for modified control limits.

For the specification limits ·7450″ and ·7550″ and $\sigma = \cdot 0014''$, the modified control limits for the mean chart are ·7521″ and ·7479″ for n = 4 and ·7520″ and ·7480″ for n = 5.

For the range chart the difference between upper and lower specification limits is taken as the upper control limit. The upper and lower specification limits are taken as L_1 and S_1 for the L-S chart. On this basis, $P_1 \times P_2$ and P_3 are given in Table III for n = 4 and 5 for varying values of m and σ.

TABLE III

Probability of accepting samples with $\bar{X}$ and R charts and L-S chart with modified Control limits.

n	m	σ	P_1	P_2	P_1P_2	P_3
4	·7500	·0014	·997	1·000	·997	·999
		·0016	·991	·999	·991	·993
		·0018	·980	·999	·980	·978
		·0020	·964	·998	·962	·951
		·0022	·944	·993	·937	·910
	·7505	·0014	·989	1·000	·989	·997
	·7510	·0014	·942	1·000	·942	·991
	·7515	·0014	·805	1·000	·805	·975
	·7520	·0014	·556	1·000	·556	·937
	·7525	·0014	·284	1·000	·284	·861
5	·7500	·0014	·998	1·000	·999	·998
	,,	·0016	·995	·999	·995	·991
	,,	·0018	·988	·999	·987	·973
	,,	·0020	·976	·996	·972	·939
	,,	·0022	·959	·989	·948	·889
	·7505	·0014	·991	1·000	·991	·996
	·7510	·0014	·944	1·000	·944	·989
	·7515	·0014	·785	1·000	·785	·969
	·7520	·0014	·500	1·000	·500	·922
	·7525	·0014	·215	1·000	·215	·824

The above Tables and O. C. curves show that when σ varies the probability of accepting the samples does not differ much for a given m. On the other hand, if m varies the probability of accepting by the L-S chart is more than that for the $\bar{X}$ and R charts. It follows, therefore, that the L-S chart is as efficient as the $\bar{X}$ and R charts if there is no appreciable shift in the mean. If the mean shifts in quick succession then the $\bar{X}$ and R charts will be able to control production better than the L-S chart. In the case of L-S chart we shall have to take corrective steps much earlier when there is a trend either increasing or decreasing in the L-S values. Thus from the practical and economical points of view the L-S chart can be recommended for use in place of the $\bar{X}$ and R charts in the majority of cases.

The O. C. curve for the modified control limits is given in Fig. 3.

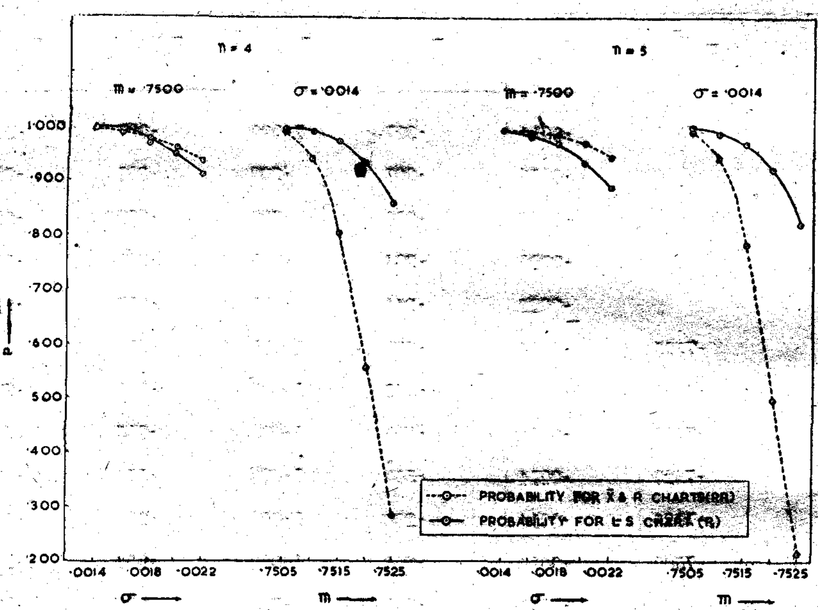

FIG 3

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