Defence Science Journal, Vol 50, No 2, April 2000, pp. 147-153

Creep Transition in a Thin Rotating Disc of Variable Density

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ABSTRACT

Creep stresses and strain rates have been obtained for a thin rotating disc having variable density using Seth's transition theory. It has been observed that a disc whose density decreases radially, rotates at higher angular speed, thus increasing the possibility of a fracture at the bore, whereas for a disc whose density increases radially, recedes the possibility of a fracture. The deformation is significant for a disc having variable density and rotating at higher angular speed.

NOMENCLATURE

e_{ii}^A	Principal finite strain component
a, b	Internal and external radii of the disc
ω	Angular velocity of rotation
u, v, w	Displacement components
r, θ, z	Radial, circumferential and axial directions
e_{ij}, T_{ij}	Strain and stress tensors
δ_{ij}	Kronecker's delta
ρ	Density of the disc
$\dot{e}_{ij}$	Strain rate tensor
ε_{ij}	Swainger strain measure
E	Young's modulus
Ω^2	$\rho\omega^2b^2/E$ (speed factor); $R=r/b$; $R_o=a/b$
σ_r	Radial stress component (T_{rr}/E)
σ_θ	Circumferential stress component $(T_{\theta\theta}/E)$

1. INTRODUCTION

Rotating discs form an essential part of the design of rotating machinery, namely, turbines, compressors, flywheels, etc. The use of rotating disc in machinery and structural applications has generated considerable interest in the solid mechanics domain. Solutions for thin isotropic discs are available in literature[1-4]. Reddy and Srinath[5] investigated the influence of material density on the stresses and displacements of a rotating disc. It has been shown that the existence of density gradient in a rotating disc influences the stresses and displacements significantly. Chang[6] has developed a closed-form elastic solution for an anistotropic rotating disc with variable density. Wahl[7] has obtained creep stresses in a rotating disc by assuming small deformation, incompressibility condition, Tresca's yield condition, a power strain law and its associated flow rule. Seth's transition theory[8] does not require these assumptions and thus solves a more general problem, from which cases pertaining to the above assumptions can be worked out. This theory utilises the concept of generalised strain measure and asymptotic solution at the critical points of the differential equations defining the deformed field. It has been successfully applied to several problems[9-12].

Seth[13] defined the generalised principal strain measure as

Revised 01 June 1999

$$\int [1 \quad 2e_{ii}^A]^{(n/2)-1} de_{ii}^A \qquad [1 \quad (1-2e_{ii}^A)^{n/2}] \tag{1}$$

where n is the measure (i ,2,3)

In this paper, the creep stresses and strain rates for a thin rotating disc of variable density have been calculated using Seth's transition theory. The density of the disc is assumed to vary along the radius in the form as

$$\rho_0 (r/b)^{-m} \tag{2}$$

where ρ_0 is the density at $r = b$, and m is the density parameter. The results obtained have been presented numerically and graphically.

2 GOVERNING EQUATIONS

Consider a thin disc of variable density with a central bore of radius a and external radius b. The disc is rotating with an angular velocity (ω) of gradually increasing magnitude about an axis perpendicular to its plane and passing through the centre. The thickness of the disc is assumed to be constant and sufficiently small, so that the disc is effectively in a state of plane stress, i.e., the axial stress T_{zz} is zero. The components of displacement in cylindrical coordinates[13] are given by:

$$u \quad r \quad \beta), \; v \quad 0, \; w \quad dz \tag{3}$$

where β is a function of $r = (x^2+y^2)^{1/2}$ only and d is a constant. The finite components of strain are:

$$\begin{aligned} e_{rr}^A &= \frac{1}{2}[1 \quad (r\beta' + \beta)^2] \\ e_{\theta\theta}^A &= \frac{1}{2}[1 \quad \beta^2] \\ e_{zz}^A &= \frac{1}{2}[1 \quad (1 \quad d)^2] \\ e_{r\theta}^A &= e_{\theta z}^A = e_{zr}^A = 0 \end{aligned} \tag{4}$$

where $\beta' = d\beta/dr$

Substituting Eqn (4) in Eqn (1), the generalised components of strain are:

$$\begin{aligned} e_{rr} &= \tfrac{1}{n}[1-(r\beta'+\beta)^n] \\ e_{\theta\theta} &= \tfrac{1}{n}[1-\beta^n] \\ e_{zz} &= \tfrac{1}{n}[1-(1-d)^n] \\ e_{r\theta} &= e_{\theta z} = e_{zr} = 0 \end{aligned} \tag{5}$$

The stress-strain relations[14] are:

$$T_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} \qquad (i,j = 1,2,3) \tag{6}$$

where λ and μ are Lame's constants and e_{kk} is the first strain invariant. Equation (6) for this problem becomes:

$$\begin{aligned} T_{rr} &= \tfrac{2\lambda\mu}{\lambda+2\mu}[e_{rr}+e_{\theta\theta}] + 2\mu e_{rr} \\ T_{\theta\theta} &= \tfrac{2\lambda\mu}{\lambda+2\mu}[e_{rr}+e_{\theta\theta}] + 2\mu e_{\theta\theta} \\ T_{zz} &= T_{zr} = T_{r\theta} = T_{\theta z} = 0 \end{aligned} \tag{7}$$

Substituting Eqn (5) in Eqn (7), the stresses are obtained as

$$\begin{aligned} T_{rr} &= \frac{2\mu}{n}[3-2C \quad \beta^n\{1 \quad C \quad (2 \quad C)(P+1)^n\}] \\ T_{\theta\theta} &= \frac{2\mu}{n}[3-2C \quad \beta^n\{2 \quad C+(1 \quad C)(P+1)^n\}] \\ T_{zz} &= T_{zr} = T_{r\theta} = T_{\theta z} = 0 \end{aligned} \tag{8}$$

where $r\beta' = \beta P$ and $C \quad \frac{2\mu}{}$

The equations of equilibrium are all satisfied except

$$\frac{d}{dr}(rT_{rr}) \quad T_{\theta\theta} + \rho\omega^2 r^2 \quad 0 \tag{9}$$

Substituting Eqn (8) in Eqn (9), one gets a nonlinear differential equation in β as

$$(2 \quad C)n\beta^{n+1}P(P \quad 1)^{n-1}\frac{dP}{d\beta} = \frac{n\rho\omega^2 r^2}{2\mu} + \beta^n[\quad (P \quad 1)^n \quad nP\{1 \quad C+(2-C)(P+1)^n\}] \tag{10}$$

The critical points of β in Eqn (10) are $P = -1$ and $P \to \pm\infty$. The boundary conditions are:

$$T_{rr} = 0 \text{ at } r = a \text{ and } r = b. \tag{11}$$

3. SOLUTION THROUGH PRINCIPAL STRESS DIFFERENCE

It has been shown[9-12] that the asymptotic solution through the principal stress difference at the transition point $P \to -1$ leads to the creep state. The transition function R is defined as

$$R = T_{rr} - T_{\theta\theta} = \frac{2\mu\beta^n}{n}\left[1-(P+1)^n\right] \tag{12}$$

$$\frac{d}{dr}(\log R) = \frac{nP}{r\{1-(P+1)^n\}}\left[1-(P+1)^n - \beta(P+1)^{n-1}(dP/d\beta)\right] \tag{13}$$

Substituting the value of $dP/d\beta$ from Eqn (10) in Eqn (13) and taking asymptotic value $P \to -1$, one gets:

$$\frac{d}{dr}(\log R) = -$$

Asymptotic value of β as $P \to -$ is D/r, D being a constant.

Integrating Eqn (14) wrt r, one gets

$$R = T_{rr} - T_{\theta\theta} = A r^K \exp(F \tag{15}$$

where A is a constant of integration, and

$$K = -\{n(3-2C)+1\}/(2-C) \text{ and}$$
$$F = -\{n\omega^2/2\mu D^n(2-C)\}\int \rho r^{n+1} dr$$

Substituting Eqn (15) in Eqn (9), one gets

$$T_{rr} = B - A\int r^{K-1}\exp(F)dr - \omega^2\int \rho r\, dr \tag{16}$$

Using boundary conditions) in Eqn (16), one gets:

$$A = \frac{-\omega^2 \int_a^b \rho r\, dr}{\int_a^b r^{K-1}\exp(F)dr}$$

and $B = A\left[\int r^{K-1}\exp(F)dr\right]$ at $r = a$
$+\omega^2\left[\int \rho r\, dr\right]$ at $r = a$

Substituting values of A and B in Eqn (16) one gets:

$$T. \quad \frac{\omega^2 \,_a\!\int^b \rho r\, dr \,_a\!\int^r r^{K-1}\exp(F)dr}{\int^b r^{K-1}\exp(F)dr} + \omega^2 \,_a\!\int^b \rho r\, dr \tag{17}$$

From Eqns (17) and (12), one has

$$\frac{\omega^2 \,_a\!\int^b \rho r\, dr}{_a\!\int^b r^{K-1}\exp(F)dr} r^K \exp(F \tag{18}$$

Equations (17) and (18) give creep stresses for a thin rotating disc of variable density.

Substituting Eqn (2) in Eqns (17) and (18), one gets the stresses in non-dimensional form as

$$\sigma_r = A_1 \,_{R_o}\!\int^R R^{K-1}\exp(F_1)dR - \frac{\Omega^2}{2-m} R^{2-m}$$

$$\sigma_\theta \quad \sigma_r + A_1 R^K \exp(F_1) \tag{20}$$

where

$$A_1 = \frac{\Omega^2\left[1 - R_o^{2-m}\right]}{(2-m)_{R_o}\!\int^1 R^{K-1}\exp(F_1)dR}$$

$$F_1 = \frac{n(3-2C)\Omega^2 R^{n-m+2} b^n}{(2-C)^2 (n-m+2) D^n}$$

$$\frac{E}{n} = \frac{(3-2C)}{(2-C)}\frac{2\mu}{n}; \quad 2-m \neq 0 \text{ and } n-m+2 \neq 0$$

For a disc made of incompressible[9-12] material, i.e. $C \to 0$, the stresses given by Eqns (19) and (20) become:

$$\sigma_r = A_2 \int_{R_0}^{R} R^{-3(n+1)/2} \exp(F_2)\, dR - \frac{\Omega^2}{2-m}\left[R^{2-m} - R_0^{2-m}\right]$$

$$\sigma_\theta = \sigma_r + A_2 R^{-(3n+1)/2} \exp(F_2) \quad (22)$$

where

$$A_2 = \frac{\Omega^2\left[1 - R_0^{2-m}\right]}{(2-m)\int_{R_0}^{1} R^{-3(n+1)/2} \exp(F_2)\, dR}$$

and

$$F_2 = \frac{3n}{4}\frac{\Omega^2 R^{n-m+2} b^n}{(n-m+2) D^n}$$

For the disc having constant density, (i.e. $m = 0$); Eqns (21) and (22) become:

$$\sigma_r = A_3 \int_{R_0}^{R} R^{-3(n+1)/2} \exp(F_3)\, dR - \frac{\Omega^2}{2}\left[R^2 - R_0^2\right]$$

$$\sigma_\theta = \sigma_r + A_3 R^{-(3n+1)/2} \exp(F_3) \quad (24)$$

where

$$A_3 = \frac{\Omega^2\left[1 - R_0^2\right]}{2\int_{R_0}^{1} R^{-3(n+1)/2} \exp(F_3)\, dR}$$

and

$$F_3 = \frac{3n\Omega^2 R^{n+2} b^n}{4(n+2) D^n}$$

These equations are the same as obtained by Shukla[11].

The creep strain rates are calculated as follows.

When the creep sets in, the strains should be replaced by strainrates and the stress-strain relations[15] are given by:

$$\dot{e}_{ij} = \frac{3}{2}\lambda_1 S_{ij} \qquad (i, j = 1, 2, 3) \quad (25)$$

where $\dot{e}_{ij}$ is the strain rate tensor wrt flow parameter t, and S_{ij} is the stress-deviator tensor.

Differentiating Eqn (5) wrt t, one gets:

$$\dot{e}_{\theta\theta} = \beta^{n-1} \dot{\beta} \quad (26)$$

For Swainger measure (i.e. $n = 1$), Eqn (26) becomes:

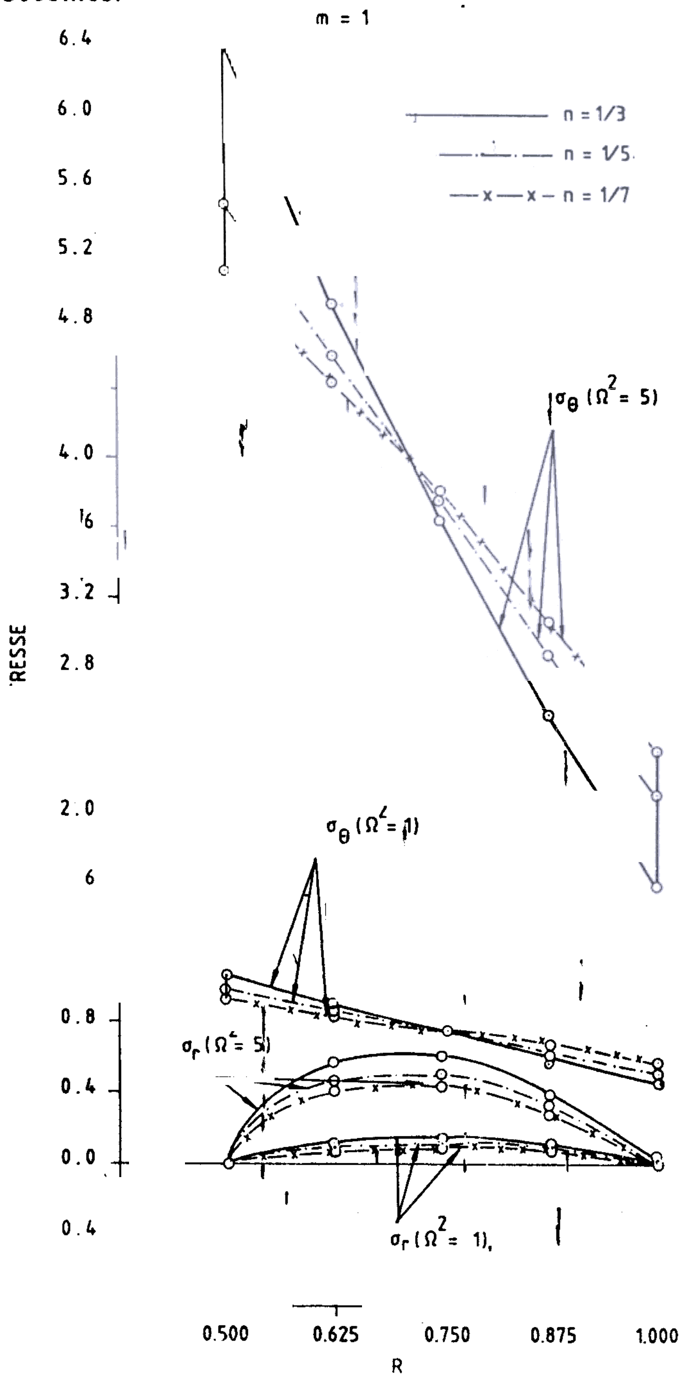

Figure 1(a). Creep stresses in a thin rotating disc having variable density ($m = 1$) along the radius.

$$\dot{\varepsilon}_{\theta\theta} = -\dot{\beta} \qquad (27)$$

where $\varepsilon_{\theta\theta}$ is the Swainger strain measure.

From Eqn (12), the transition value of β is:

$$\beta = (n/2\mu)^{1/n}\left(T_{rr} - T_{\theta\theta}\right)^{1/n} \qquad (28)$$

Substituting Eqns (26), (27) and (28) in Eqn (25), one gets:

$$\dot{\varepsilon}_{\theta\theta} = \chi\left[3n(\sigma_r - \sigma_\theta)/2\right]^{(1/n)-1}(S_{ij}/\mathrm{E}) \qquad (29)$$

where $\chi = (3/2)\lambda_1 \mathrm{E}$ is a constant and λ_1 has the same dimension as that of (1/E).

Since the form in Eqn (29) is to be valid, one must have

$$\dot{\varepsilon}_{ij} = \chi\left[3n(\sigma_r - \sigma_\theta)/2\right]^{(1/n)-1}(S_{ij}/\mathrm{E}) \qquad (30)$$

These are the constitutive equations used by Odquist[16] for finding the creep stresses, provided $n = 1/N$.

4. DISCUSSION

For calculating the stresses and strain rate distribution based on the above analysis, the following values have been taken:

$$\Omega^2 = \left(\rho_0\omega^2 b^2/\mathrm{E}\right) = 1, 5\,;\; m = -1,\ 0,$$
$$n = 1/3,\ 1/5,\ 1/7 \text{ (i.e. } N = 3,\ 5,\ 7).$$

Curves have been drawn in Figs 1(a), 1(b) and 1(c) between stresses σ_r, σ_θ and radii ratio R for a rotating disc made of incompressible material and having variable density. It has been shown in Fig. 1(a) that the circumferential stress is maximum at the internal surface of a disc (density decreases radially, m = 1) rotating with angular speed $\Omega^2 = 5$ for n = 1/3, 1/5 and 1/7, respectively. The value of these maximum circumferential stresses decreases at the internal surface of a disc having constant density (m = 0) (see Fig. 1(b)) or density

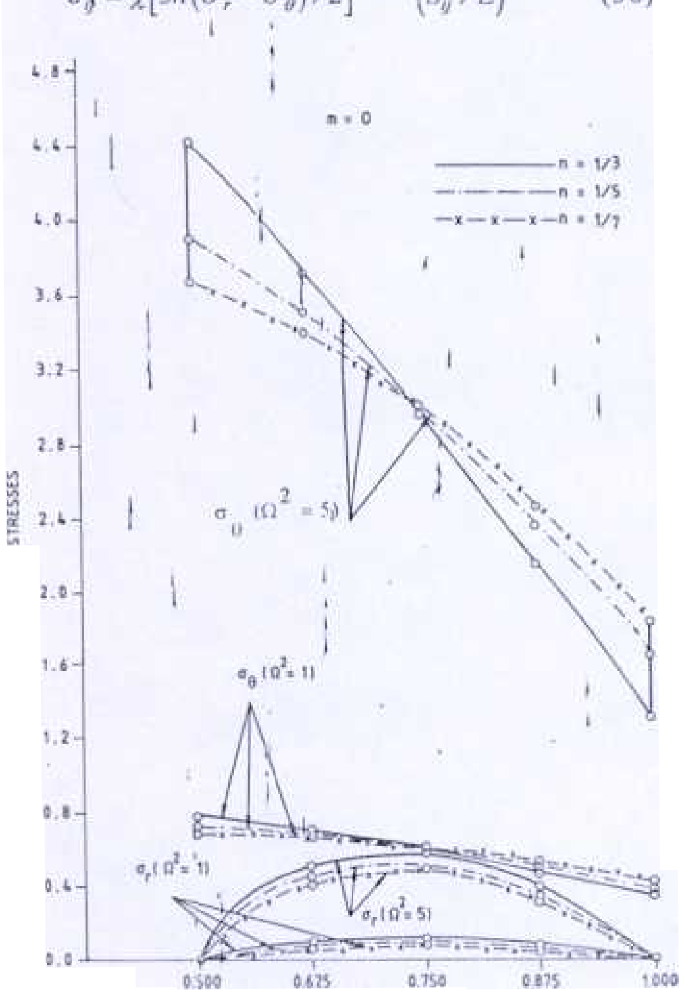

Figure 1(b). Creep stresses in a thin rotating disc having variable density (m = 0) along the radius.

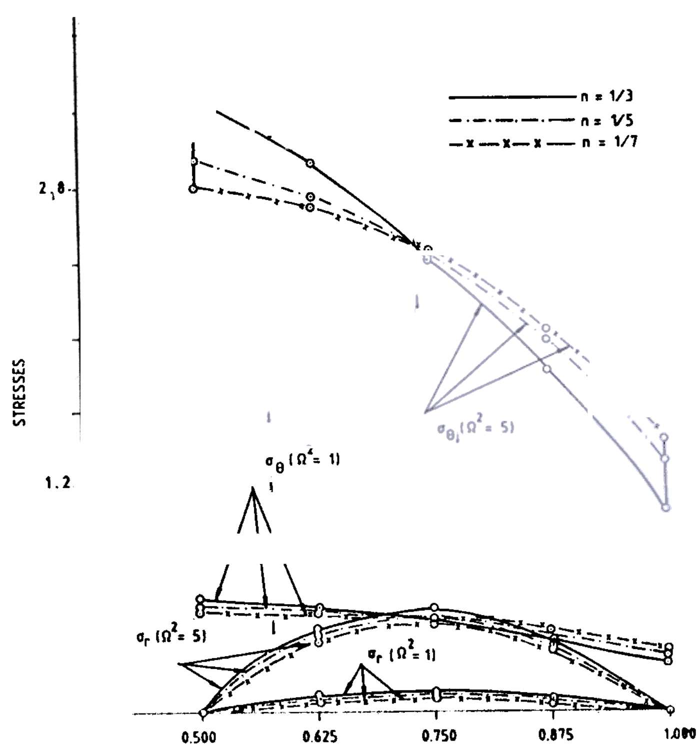

Figure 1(c). Creep stresses in a thin rotating disc having variable density (m = –1) along the radius.

increasing radially ($m = -1$) (see Fig. 1(c)). As reported by Rimrott[17], a material tends to fracture by cleavage. It is likely to begin as a sub-surface fracture close to the bore, because the largest tensile stress occurs at this location. This means that for a disc rotating with higher angular speed and whose density decreases radially, the possibility of a fracture at the bore increases, whereas for a rotating disc whose density increases radially, the possibility of a fracture at the bore decreases.

In Fig. 2, curves have been drawn between the strain rates and radii ratio R for angular speed $\Omega^2 = 1$, 5 and $n = 1/7$. It has been observed that a disc having variable density and rotating with angular speed $\Omega^2 = 1$ has negligible deformation, whereas the deformation is significant for angular speed $\Omega^2 = 5$.

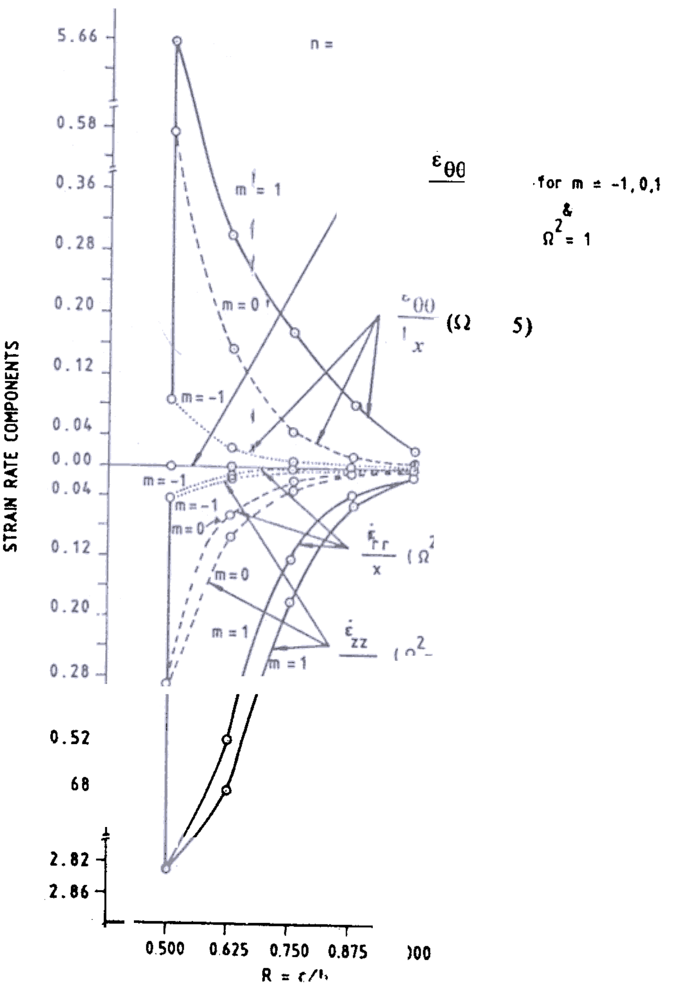

Figure 2. Strain rate components for a thin rotating disc having variable density along the radius.

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