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# Weighted Generalised Directed-Divergence Measure to Assess Military Requirements

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**ABSTRACT**

In the present paper, a weighted information theoretic measure has been used to compare and assess the military requirements of a country wrt other countries to meet the challenge of future battles. A measure of weighted directed-divergence based on m probability distributions has been proposed and a probability distribution 'closest' to these m probability distributions is obtained. The closest probability distribution provides a reasonably adequate measure and thus enables one to apply this technique in real life situation, viz., assessment of balanced military requirements for a country: consensus ranking, pattern recognition, etc.

## 1. INTRODUCTION

We are living in a world dominated by highly sophisticated weapon systems, including nuclear technologies. Different countries are either designing or procuring these weapon systems to meet future battle needs. It is equally important for our nation to compare its own military requirements in relation to other countries to meet national security needs and future military threats. The military requirements may be in terms of defence budget, quantities of different weapon systems, manpower in Armed Forces, etc. The model developed in this study is based on the principle that if the proportions of any characteristic (say weapon system) for different entities (say countries) are known, then the proportions of the characteristic 'closest' to the proportions of characteristics of different entities can be obtained. Earlier, a generalised measure of directed-divergence was proposed[1] and the same applied to assess military requirements. In the present paper, an effort has been made to incorporate a weight importance factor, since it is known that different weapon systems are applicable in different environment due to operational limitations. For example, the mobility of a tank may depend upon the terrain quality, which varies from one country to another. Therefore, the requirement of number of tanks is different for different countries. The concept of weight factor may account for the operational environment and applicability of weapon systems. Further, reliability of different weapon systems is also incorporated in the model.

## 2. GENERALISED DIRECTED-DIVERGENCE MEASURE

Let $P_1, P_2, \ldots P_m$ be $m$ probability distributions, where $P_r = (p_{1r}, p_{2r}, \ldots p_{nr})$ ; $r = 1, 2, \ldots, m$.

Vinocha and Goyal[2] obtained a measure of generalised directed-divergence based on these $m$ probability distributions as:

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$$D_1(P_1, P_2, \ldots, P_m) = \sum_{i=1}^{n} p_{i1} \ln \frac{p_{ir}}{p_{im}} - \sum_{\substack{k=2 \\ k \neq r}}^{m-1} p_{i1} \ln p_{ik} \quad (1)$$

$$= I(P_1 : P_m : P_r) - \sum_{\substack{k=2 \\ k \neq r}}^{m-1} I(P_1 : P_k) \quad (2)$$

where $I(P_1{:}P_m{:}P_r)$ is Theil's[3] measure of information improvement, when the true distribution is $P_1$ and its estimate is revised from $P_m$ to $P_r$, and $I(P_1{:}P_k)$ is a Kerridge[4] measure of inaccuracy. This measure was studied by Kapur and Tripathi[5] and it has been found that this measure is a characterisation of a specific functional equation and does not satisfy the essential properties of measures of information. Also, it was modified, so that it can become a meaningful measure. Kapur[6] proposed other generalised directed-divergence measures based on $m$ probability distribution.

Equation (1) can also be written as

$$D_1(P_1, P_2, \ldots, P_m) = (m-2) \sum^{n} p_{i1} \ln \frac{p_i^*}{(p_{im})^{1/m-2}} \quad (3)$$

where $p_i^*$ is geometric mean of the $i^{th}$ components of $P_2, P_3, \ldots P_{m-1}$ and

$$\sum_{i=1}^{n} p_i^* = A \quad (4)$$

so that

$$P^* = \left( \frac{p_1^*}{A}, \frac{p_2^*}{A}, \ldots, \frac{p_n^*}{A} \right) \quad (5)$$

is the probability distribution.

## 3. INTERPRETATION OF CLOSEST PROBABILITY DISTRIBUTION

The sum of Kullback-Leibler[7] measure of directed-divergence of a probability distribution $P$ from $P_2, P_3, \ldots P_{m-1}$ is given by

$$D_2\{P:(P_2, \ldots, P_{m-1})\} = \sum_{r=2}^{m-1} D(P:P_r)$$

$$= \sum_{i=1}^{n} \sum_{r=2}^{m-1} p_i \ln \frac{p_i}{p_{ir}} \quad (6)$$

$$= (m-2) \left[ \sum_{i=1}^{n} p_i \ln \frac{p_i}{p_i^*/A} - \ln A \right] \quad (7)$$

This is minimum when $P = P^*$, where $A$ and $P^*$ are given by Eqns 4 and 5, respectively. Thus, $P^*$ is the distribution, the sum of directed-divergence from which to $(m-2)$ distribution $P_2, P_3, \ldots P_{m-1}$ is minimum. It is, in some sense, the distribution which is closest to the $(m-2)$ probability distributions.

## 4. WEIGHTED INFORMATION-THEORETIC MEASURE

Literature on measures of information and their characterisation has been surveyed by Aczel and Daroczy[8] and Mathai and Rathie[9]. They have discussed measures of entropy, directed-divergence, information-improvement, but have not discussed the useful measures (weighted measures). Belis and Guiasu[10] raised the important issue of integrating the quantitative, objective and probabilistic concepts of information with the qualitative subjective and non-stochastic concept of utility (weight). On the basis of two plausible postulates, viz., (i) the useful/weighted information from two independent events is the sum of the useful/weighted information given by the two events separately and the (ii) weighted information given by an event is directly proportional to its weight, one can deduce

$$H(P{:}W) = -\sum_{i=1}^{n} w_i p_i \ln p_i \quad (8)$$

where the utility/weight distribution is $W = (w_1, w_2, \ldots w_n)$ and the probability distribution, $P = (p_1, p_2, \ldots p_n)$.

Longo[11] discussed this measure in detail and raised two objections against its applicability in noiseless coding theorem. Kapur[12] further studied $H(P{:}W)$ in depth and raised two more objections regarding the unit of information; and he could modified this measure, so that it can become a meaningful measure of information. As utility/weight increases, the information does not change, but the expected utility (weight) increases:

$$J(P) = \sum^{n} p_i w_i$$

where $J(P)$ is the expected utility. He has also given a justification to couple the weight (utility) concept with information.

If one defines

$$P_i' = \frac{w_i p_i}{\sum_{i=1}^{n} w_i p_i} \tag{9}$$

then $-\sum_{i=1}^{n} P_i' \ln P_i'$ becomes a proper measure of information and satisfies all the important properties. Kapur[13] further proposed a number of useful measures of information. The inclusion of $w_i$'s in the measures of information enables one to extend the scope of application.

## 5. WEIGHTED MEASURE OF GENERALISED DIRECTED-DIVERGENCE

Let $P_j = (p_{1j}, p_{2j}, \ldots, p_{nj})$: $j = 1,2, \ldots, m$ be $m$ proportion vectors having $n$ components each or probability distributions and $W_j = (w_{1j}, w_{2j}, \ldots w_{nj})$ are weight factors assigned to each component of $m$ proportion vectors, $(0 \le w_{ij} \le 1)$, $\sum_{i=1}^{n} w_{ij} = 1$, $j = 1,2,\ldots, m$.

Let $P_1$ be the true proportion vector and let its initial estimate be $P_m$. Let $P_2, P_3, \ldots P_{m-1}$ be any $(m-2)$ revised estimate of $P_1$; then the $(m-2)$ improvements of information are:

$$I(P_1 : P_m : P_2),\ I(P_1 : P_m : P_3),\quad I(P_1 : P_m : P_m \tag{10}$$

and the average of these is given by

$$D_2(P_1, P_2, \ldots, P_m) = \frac{1}{m-2} \sum_{i=1}^{n} \sum_{r=2}^{m-1} \left( \frac{w_{i1} p_{i1}}{\sum_{i=1}^{n} w_{i1} p_{i1}} \right) \ln \frac{\left( \frac{w_{ir} p_{ir}}{\sum_{i=1}^{n} w_{ir} p_{ir}} \right)}{\left( \frac{w_{im} p_{im}}{\sum_{i=1}^{n} w_{im} p_{im}} \right)}$$

$$= \frac{1}{(m-2)} \sum_{i=1}^{n} \sum_{r=2}^{m-1} P_{i1}' \ln \frac{P_{ir}'}{P_{im}} \tag{12}$$

where

$$P_{ik}' = \left( \frac{w_{ik} p_{ik}}{\sum_{i=1}^{n} w_{ik} p_{ik}} \right), \qquad k = 1,2,\ldots, m$$

$$= \frac{1}{(m-2)} \sum_{i=1}^{n} P_{i1}' \ln \frac{P_{i2}' . P_{i3}' \ldots . P_{i(m-1)}'}{P_{im}'^{(m-2)}}$$

$$= \frac{1}{(m-2)} \sum_{i=1}^{n} P_{i1}' \ln \frac{P_i'^{*(m-2)}}{P_{im}'^{(m-2)}}$$

$$= \sum_{i=1}^{n} P_{i1}' \ln \frac{P_i'^{*} / A'}{P_{im}'} + \ln A' \tag{16}$$

$$I(P_i' : P_m' : P'^{*}) + \ln A'$$

where $P_i'^{*}$ is a weighted geometric mean of the $i^{th}$ components of $P_2', P_3', \ldots P_{m-1}'$

and

$$A' = \sum_{i=1}^{n} p_i'^{*} \tag{18)}$$

so that

$$P'^{*} = \left( \frac{P_1'^{*}}{A'}, \frac{P_2'^{*}}{A'}, \quad \frac{P_m'^{*}}{A'} \right) \tag{19}$$

is the probability distribution

It has meaningful interpretation in terms of improvement of information in revising the estimate of $P_1'$ from $P_m'$ to $P'^{*}$ where $P'^{*}$ is the distribution closest to $P_2'$, $P_3' \ldots P_{m-1}'$. The normalisation in Eqn (11) is necessary for $P_{i1}'$, $P_{i2}', \ldots, P_{im}'$ to be a probability distribution. Instead of simple weight factor, one could also consider any monotic increasing function of weight. This weight function may depend on several parameters which characterise the utility of a weapon system and the choice of these parameters is up to the designer or the expert.

## 6. APPLICATIONS OF PROPOSED MODEL FOR ASSESSING MILITARY REQUIREMENTS

The measures developed in this study may be useful in various scenarios. Some of them are as follows:

- Let $p_{ij}$ ($i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$) in Eqn (11) represent the proportion of defence budget allocated to $j^{th}$ category of weapons (say missiles, tanks, torpedo etc.) by $i^{th}$ country and $w_{ij}$ be the weight factor assigned to $j^{th}$ category of weapons by $i^{th}$ country. These weights are proportional to the operational environment and the applicability of a particular weapon system. In general, $p_{ij}$'s and $w_{ij}$'s are different for different countries. $P'^{*}$s, in Eqn 16 so obtained may be the appropriate choice, i.e., at least the country should have the proportion of defence budget allocated to different categories of weapon systems which is closest to the given proportions of defence budgets allocated to different categories of weapon systems by other countries. Different components of $P'^{*}$ give the proportions of defence budget allocated to various categories of weapon systems.

- Let $p_{ij}$ ($i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$) in Eqn (11) represent the proportions of different categories of weapon systems and $w_{ij}$ have the same interpretation as above, then $P'^{*}$ in Eqn (16) gives the proportions of different categories of weapon systems which are closest to the given proportions of these categories of weapon systems for other countries.

- As one knows, reliability plays an important role in design and development of hardware systems[14]. This is the probability of a device to perform its purpose adequately for an intended time (mission) under given environmental conditions. Coupling of reliability factor with this model may give integrated proportions.

Let $r_{ij}$ ($0 \le r_{ij} \le 1, i = 1, 2, \ldots n; j = 1, 2, \ldots m$) be the reliability factor assigned to $j^{th}$ category of weapons by $i^{th}$ country. $p_{ij}$ and $w_{ij}$ are the same as in the above application.

One defines

$$P_{ik}'' = \frac{r_{ik} w_{ik} p_{ik}}{\sum^{n} r_{ik} w_{ik} p_{ik}} \qquad k = 1, 2, \quad , m$$

One gets:

$$= D_3 (P_1'', P_2'', \quad P_m'') = \sum_{i=1}^{n} P_{i1}'' \ln \frac{P_i''^{*} / A''}{P_{im}''} + \ln A'' \tag{21}$$

$$= I (P_1'' : P_m'' : P''^{*} \quad + \ln A'' \tag{22}$$

$$P''^{*} = \left( \frac{P_1''^{*}}{A''}, \frac{P_2''^{*}}{A''}, \quad \frac{P_m''^{*}}{A''} \right)$$

$$A'' = \sum_{i=1}^{n} p_i''^{*} \tag{24}$$

$P''^{*}$ is a proportion of different categories of weapon systems coupled with reliability and weight factor, which is closest to the given proportion of various categories of weapon systems of other countries. In this way, one can obtain a proportion of different categories of weapon systems along with their reliabilities and utilities, which at least a country should have to meet the challenge of future battles. One could also consider these proportions for various parameters measuring the military strength of different countries. $w_{ij}$ ($0 \leq w_{ij} \leq 1$) depends upon applicability and operational environment, etc. of these parameters; this procedure provides a balanced military requirement for a country.

- Suppose there are different categories of communication, say, open friendly, covertly conciliatory, indifferent, covertly hostile and openly threatening. Let $p_{ij}$ be the proportion of $j^{th}$ category of messages communicated by $i^{th}$ country. In this case, a measure of directed-divergence of one country from another or from a group of countries is needed in order to measure an external threat perception[15]; such a measure has been precisely developed in this study. After the nuclear testing in India, different countries are giving their statements on this subject. The counting and classification of these statements need to be aided by a group of political, economic, diplomatic, strategic and service experts. One could also assign $w_{ij}$, an authoniticity factor, to each $p_{ij}$. In this scenario, such measure may help to judge the reactions of different countries towards India, or, in general, it may help one to know the interaction among various countries.

- This methodology could also be used for measuring the difference of opinions among groups of persons, in ranking and selection and in pattern recognition, etc. with a proper interpretation of weight/utility factor according to the situation being encountered.

## 7. DISCUSSION & CONCLUSIONS

The measure developed in this study is obtained from the average of $(m-2)$ improvements of information. One could also consider groups of probability distributions and obtain new measures by reducing these into three groups of probability distributions. Instead of taking arithmetic mean of these $(m-2)$ information improvements, one could take some other central tendency measure. A number of parametric measures of information are available in literature[16]. One could also consider them and apply the same methodology to get more generalised directed-divergence measures coupled with reliability and weight factor. Most of these parameteric measures tend to Kullback-Leibler measure of directed-divergence for specific values of the involved parameters. The role of involved parameters in these measures gives the accountability of the goodness of fit.

The information required to apply these measures to assess military requirements includes the proportions of defence budget, different categories of weapon systems, their applicability and the operational environment (i.e., weightage factor), reliability figures of different categories of weapon systems of various countries which may be available, but it may be difficult to get the exact figures for other countries.

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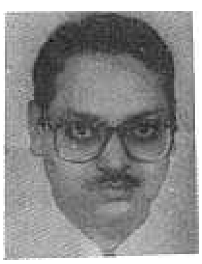

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