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Resonant Frequencies of Whispering Gallery Modes of Dielectric Resonator

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ABSTRACT

The modal spectrum of the whispering gallery modes of dielectric resonator depends mainly on its physical dimensions, dielectric constant, and to a lesser extent, on the environment. This paper carries investigation of the resonant frequencies in dielectric disc utilising the ring resonator model. Results of the structural design parameters are used to generate a numerical expression for describing the operational frequencies useful for computer-aided design applications. Theoretical and experimental results are given for resonators of various sizes.

Keywords: Whispering gallery mode, dielectric resonators, ring resonator model, resonant frequencies

1. INTRODUCTION

The typical applications of dielectric resonators (DRs) are in microwave stable fixed-frequency oscillators, narrow band filters, etc. The use of DRs offer advantages of high efficiency, high temperature stability and easy integrability with hybrid microwave integrated circuits (MICs). However, the use of conventional DRs in $TE_{01\delta}$ mode becomes difficult, especially at millimeter waves as dimensions of DR become too small to be easily realised and handled. The planar circuit whispering gallery (WG) modes of DR are a promising solution [1,2,3]. Cylindrical DRs when used in WG mode become oversized, offer distinct advantages, such as high quality factor and quasi insensitivity to conducting material boundaries. Resonant frequencies and loaded quality factors are critical to any practical applications like DR oscillators and DR filters. Properties of WG modes of DRs are invaluable for successful design of resonators, both when these are used in design or when they are potentially excited and interfere as spurious modes.

In this paper, numerical analysis is carried out to generate data for describing operational resonant wave numbers for various azimuthal mode indexes. The self-contained program developed using commercial mathematical software provides resonant wave numbers through solution of coupled pair of equations. The data obtained from various structural design parameters, like physical dimensions of the disc resonator and its dielectric constant are used to generate a numerical expression for describing resonant frequencies. Theoretical, numerical and experimental results are given for resonators of various sizes.

2. WHISPERING GALLERY MODES

Whispering gallery modes move essentially in the plane of the circular cross-section, and most

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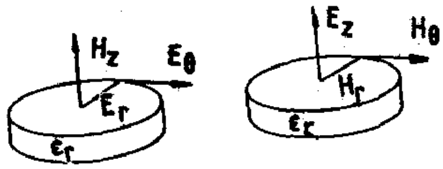

Figure 1. Electromagnetic fields of WGE and WGH modes

of the modal energy is confined between the resonator boundary and the inner modal caustic as well as to within a small region in an axial direction. WG mode disc DR is thus capable of achieving high quality factor due to circumferential modes isolating electromagnetic energy into a narrow dielectric element itself and away from lossy environment.

WG modes are classified as either $WGE_{n,m,l}$ or $WGH_{n,m,l}$. As shown in Fig.1 for WGE modes, the electric field is essentially transverse and for WGE modes, the electric field is axial. Here, the integer n denotes the azimuthal variation of modes, m the radial variation and l the axial ones. WGE modes are of interest as they magnetically couple with the quasi TEM mode of microstrip as shown in Fig. 2.

2.1 Electromagnetic Fields of WGE DR Modes

For a cylindrical geometry of a DR of high relative permittivity and height not too small, the perturbational effect due to composite structure is minimal and frequency computation can be done using idealised travelling ring resonator as depicted in Fig. 3. Cylindrical DR has a radius a and thickness H. The semi-numerical approach that is followed involves idealisation of the configuration as a ring resonator structure with external radius as the resonator

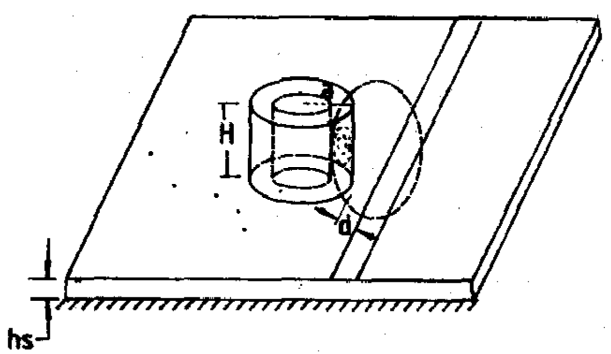

Figure 2. Cylindrical dielectric resonator on microstrip line

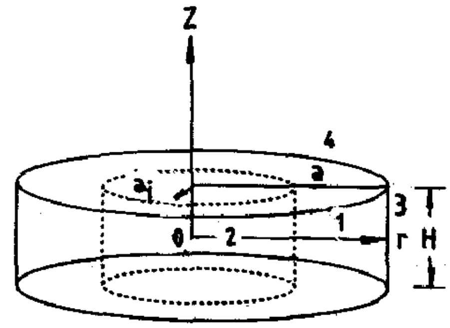

Figure 3. Dielectric ring resonator model

physical boundary, while the imaginary internal one being radius of a model caustic[4]. Most of the energy is confined in region 1, between the cylindrical boundary, a and the inner modal caustic, a_i. Analytically, it means that the solution of the Bessel equation in the part of the space inside the dielectric disc is oscillating for $r > a_i$ and monotonically decreasing for $r < a_i$ (under asymtotic approximation Bessel functions can be replaced by Airy functions) and the field decays exponentially in regions 3 and 4. Electromagnetic fields are written in partial regions of the resonator. Resonant frequencies are computed by solving a boundary value problem for this ring resonator structure.

The electric and magnetic fields satisfy Maxwell equations. However, rather than solving the coupled set of equations, both E and H are solved independently so that for WGE modes, the longitudinal magnetic field components in each sub-region of the structure can be obtained by solving appropriate Helmholtz equation[2,5,6]:

$$\nabla^2 H_{zi} + k^2 H_{zi} = 0 \qquad i = 1,2,3,4 \tag{1}$$

The expressions of the longitudinal magnetic field components in each zone of the structure are given as

$$H_{z1} = \left[AJ_n(k_1 r) + BY_n(k_1 r) \right] \cos(\beta z) \exp(jn\theta) \tag{2}$$

$$H_{z2} = EI_n(k_2 r) \cos(\beta z) \exp(jn\theta) \tag{3}$$

$$H_{z3} = HK_n(k_3 r) \cos(\beta z) \exp(jn\theta) \tag{4}$$

$$H_{z4} = -N\left[AJ_n(k_1r) + BY_n(k_1r)\right] \exp(-\alpha z)\exp(jn\theta) \quad (5)$$

where

$$k_0 = \omega/c \quad (6)$$

$$k_1^2 = k_0^2\varepsilon_{r1} - \beta^2 \quad (7)$$

$$k_2^2 = k_0^2\varepsilon_{r2} - \beta^2 \quad (8)$$

$$k_3^2 = \beta^2 - k_0^2 \quad (9)$$

$$\alpha^2 = k_1^2 - k_0^2 \quad (10)$$

and

$$a_i = n/k_1 \quad (11)$$

Here, k_0 is the free space wave number, k_1 and k_2 are electromagnetic mode separation constants in dielectric regions 1 and 2, respectively, c is the velocity of the light, α is attenuation constant and β is axial propagation constant. The solution of the Maxwell equations can be employed to obtain transverse components E_r, E_θ, H_r and H_θ as functions of longitudinal component H_z.

2.2 Boundary & Continuity Conditions for Resonant Frequencies

From the above expressions for H longitudinal field components, a complete set of electromagnetic field expressions can be obtained. First, the tangential components at $r = a_i$ and $r = a$ are matched so that

$$E_{\theta 1} = E_{\theta 2}$$

$$H_{z1} = H_{z2}$$

and

$$E_{\theta 1} = E_{\theta 3}$$

$$H_{z1} = H_{z3}$$

Using these, a set of four equations is obtained. For a non-trivial solution, the determinant of the resulting 4×4 matrix should vanish. This condition is equivalent to

$$\begin{vmatrix} J_n(k_1a_i) & Y_n(k_1a_i) & -I_n(k_2a_i) & 0 \\ k_2J'_n(k_1a_i) & k_2Y'_n(k_1a_i) & -k_1I'_n(k_2a_i) & 0 \\ J_n(k_1a) & Y_n(k_1a) & 0 & -K_n(k_3a) \\ k_3J'_n(k_1a) & k_3Y'_n(k_1a) & 0 & -k_1K'_n(k_3a) \end{vmatrix} = 0 \quad (12)$$

Here, J_n, Y_n are Bessel functions of first kind and of order n. I_n and K_n are modified Bessel functions of second kind and order n. The prime denotes the differentiation of a function wrt its argument. To obtain correct resonant frequency, another set of equations is obtained, by forcing axial confinement at dielectric air interface on fields, through boundary condition of continuity and their first-order derivatives. Considering imperfect magnetic wall separating air dielectric region at $z = H/2$ and $-H/2$, so that

$$E_{r1} = E_{r4}$$

$$\frac{\partial E_{r1}}{\partial z} = \frac{\partial E_{r4}}{\partial z}$$

on simplification, this reduces to a characteristic equation given as

$$\tan\frac{\beta H}{2} = \frac{\alpha}{\beta} \quad (13)$$

$$\alpha^2 = k_0^2(\varepsilon_{r1} - 1) - \beta^2$$

which is a characteristic equation for the TE_0 mode of a radial slab guide having the same ε_r and height, H as that of the resonator.

2.3 Numerical Implementation

It is generally known that the above form of equations do not permit analytical solution. So, numerical approach is followed to obtain desired solution. The resonant frequency is obtained by simultaneous solution of two equations. It is observed that Eqn (12) relates two unknowns, one being the free space wave number k_0 and the other axial propagation constant β. All other parameters can be found out through k_0 and β. Equation (13) relates perturbational effect on resonant frequency due to axial propagation consideration for a finite thickness of the sample. This equation was chosen for numerical implementation through root search algorithm developed[7]. The form of equation used is given by

$$f(k_0,\beta)=f([k_i,k_f,k_s],\beta_i)=0$$

For more reliable behaviour, the lower bound k_i is decided *a priori*[1]. The upper bound which is not a restriction is a typically $2k_i$ sufficient for covering lower azimuthal mode index resonance frequencies. The testing interval k_s is typically 5, sufficient for unambiguous identification of sign reversal region of the determinant. It has been convincingly established that for a specified range of k_0, β values are always unique irrespective of its starting value β_i. With the array of $[k_0, \beta]$ the Eqn (12) is tested using a similar root search algorithm for a particular azimuthal mode index.

$$g(k_0,\beta)=g([k_i,k_f,k_s],[\beta_i,\beta_f],n)=0$$

3. NUMERICAL EXPRESSION FOR RESONANT FREQUENCIES

The above program was run for a set of various Tran-Tech 8700 series resonator dimensions and numerical data was generated. This data has been used to generate a polynomial expression using standard numerical technique. The derived numerical expression for the resonant frequency is given by

$$f_0=(1/D)[131.18-697.037\,(H/D)+2060.996\,(H/D)^2-2138.93\,(H/D)^3+27.233(n)+7.088(n)\,(H/D)-12.78(n)\,(H/D)^2-1.142\,n^2-0.306\,n^2\,(H/D)+0.069\,n^3]$$

where $D = 2a$ is the diameter of the resonator and n is the azimuthal mode index.

It can be observed that first term is a constant for a series. Next three terms give the perturbation of f_0 due to finite thickness of the sample. It can also be noted that for a particular height of a resonator, f_0 increases rapidly with mode index. Higher order terms contribute much less to the perturbation of f_0.

The generated expression is accurate to predict resonant frequencies over height-to-diameter ratio of 0.1365 to 0.3412 and for azimuthal mode index up to 6. The dielectric constant assumed for this series is 30 and the range of frequency is from 10 GHz to 40 GHz.

4. COMPUTED & EXPERIMENTAL RESULTS

Resonant frequencies were computed using Eqns (12) and (13). An experiment was carried out to find the resonant frequency. As shown in Fig. 2, a 50 ohm line was printed on RT-duroid-5880 substrate of 0.127 mm (*hs*) and die attached to the ground block.The DR was kept by the side of the transmission line and the distance *d* of the resonator edge from the transmission line varied, thus affecting the transmission characteristics.The DR coupled to the microstrip line acts as a band-stop filter. The resonant frequencies were measured using HP-8757C network analyser. The theoretical, numerical and experimental results are shown in Tables 1 and 2. Figure 4 shows modal spectrum for a WG DR as displayed on a network analyser corresponding to two modes in Table 1.

5. DISCUSSION & CONCLUSION

As can be seen, there is in general a close agreement between the theoretical and experimental results.The simple numerical expression is given for computation of resonant frequency for different aspect ratios and mode index. The expression is

Table 1. Theoretical, numerical and experimental results

Modes	Results		
	f_0 (GHz) Theoretical	f_0(GHz) Numerical	f_0(GHz) Experimental
$WGE_{3,0,0}$	17.57	17.57	19.21
$WGE_{4,0,0}$	20.62	20.65	21.59
$WGE_{5,0,0}$	23.53	23.61	24.02

Resonator–Trans-Tech-8700
ε_{r1}= 30.00, 2a = 7.23 mm
H = 1.90 mm

Substrate–RT-duroid-5880
ε = 2.20, *hs* = 0.127 mm
W = 0.38 mm

Table 2. Theoretical, numerical and experimental results

Modes	Results		
	f_0 (GHz) Theoretical	f_0(GHz) Numerical	f_0(GHz) Experimental
$WGE_{3,0,0}$	12.89	12.94	13.30
$WGE_{4,0,0}$	15.27	15.10	15.06
$WGE_{5,0,0}$	17.19	17.19	17.70
$WGE_{6,00}$	19.09	19.25	19.57

Resonator–Trans-Tech-8700
ε_{r1} = 30.00, 2a = 10.30 mm
H = 2.00 mm

Substrate–RT-duroid-5880
ε_s = 2.20, hs = 0.127 mm
W = 0.38 mm

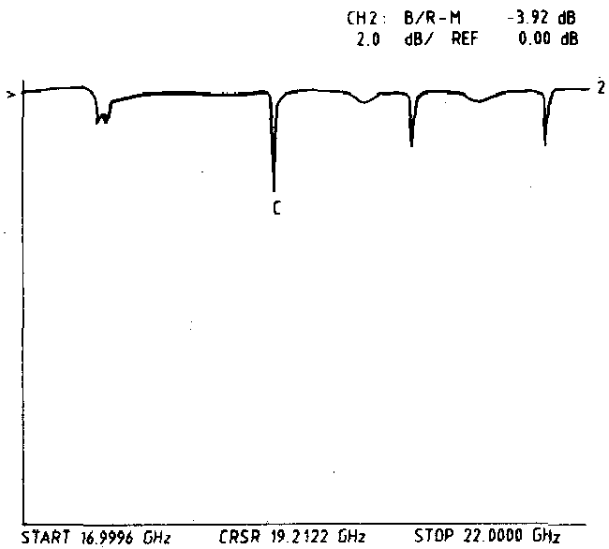

Figure 4. Modal spectrum for whispering gallery mode dielectric resonator.

reasonably accurate over the specified bound. When thickness of the DR sample matters wrt substrate thickness and effect of the substrate parameters has to be investigated, this expression will serve purpose of initial guess for wave number computation.

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