Punch Problem for Initially Stressed Neo-Hookean Solids

M. Mukhtar Ali, Gh. Nabi Parrey and Rashid Ali
Aligarh Muslim University, Aligarh - 202 002

ABSTRACT

The paper reports the indentation of a semi-infinite, initially stressed elastic medium under the action of an axisymmetric rigid punch pressing the medium normally. The problem has been considered within the framework of incremental deformation theory for neo-Hookean solids. Using the Hankel's transformation, the distributions of incremental stress and strain have been obtained. Indentations by a flat-ended circular cylindrical punch and a conical punch have been obtained as special cases and these effects have been studied numerically and presented in the form of curves. *This problem has defence application as many launching pads and firing machines have some* neo-Hookean solids as buffer which bear enormous impact or punch during the action of the machine.

Keywords: Neo-Hookean solids, Hankel's transformation, incremental deformation theory, axisymmetric rigid punch, cylindrical punch, conical punch

NOMENCLATURE

x_i	Cartesian coordinates
n_i	Components of unit normal to boundary surface
S_{ij}	Initial stress, corresponding to initial finite deformation referred to x_i
ρ	Density in a finite deformation
W	Elastic potential per unit volume
u_i	Incremental displacement (infinitesimal)
λ_i	Extension ratio
e_{ij}	Incremental strain
ϕ	Displacement function
w_{ij}	Incremental rotation
e	Incremental volume expansion
s_{ij}	Incremental stress referred to axes which are incrementally displaced with the medium
Δf_i	Incremental boundary force per unit initial area
μ_o	Shear modulus in an unstrained state
P	Initial all-around compressive stress
$A(\xi)$ $B(\xi)$	Integral constants

1. INTRODUCTION

Various elastic bodies posses initial stress which exists in the body by process of preparation or by the action of body forces, e.g. a sheet of metal rolled up into a cylinder and the edges welded together. If such a body is further subjected to deforming forces then apart from the initial finite deformation, it will have incremental deformation also. Many tanks and missile firing machines have some buffer material to bear the impact of reverse action of the machine when they are fired. In these cases, the effect of punch on the machine can be studied in the light of findings of this paper. Trefftz[1], Neuber[2], Green[3,4] and Biot[5,6] have discussed and

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given basic equations of such incremental deformation theory. The derivation of basic equations generally comes from the theory of finite deformation making use of tensor calculus. But Biot has developed his theory using Cartesian concepts and elementary mathematical method. Using his theory (in which it is not only easy to understand the physical meaning of incremental stress and strain but also useful for mathematical analysis), he has treated some interesting problems. Later on, Kurashige[7,8] discussed an axisymmetric circular crack problem and a two-dimensional crack problem for an initially stressed neo-Hookean solid. The problem of opening of a crack of prescribed shape in an initially stressed body has been discussed by Ali[9]. Hara[10], *et al.* Invove[11], *et al.* and Sakamoto[12], *et al.* have discussed some contact problems. Recently, Fan and Hwu[13] have discussed punch problems for an isotropic elastric half-plane by combining Stork's formalism and the method of analytic continuation.

In the present study, an attempt has been made to find the stresses and strains in a semi-infinite, initially stressed elastic medium, which is pressed normally by an axisymmetric rigid punch. The medium is supposed to be isotropic, homogenous and incompressible.

2. BASIC EQUATIONS

In the rectangular Cartesian coordinates, (x_i and time, t), the equations of motion for incremental deformation theory and the expressions of incremental boundary forces per unit area are:

$$\frac{\partial s_{ij}}{\partial x_j}+S_{jk}\frac{\partial w_{ik}}{\partial x_j}+S_{ik}\frac{\partial w_{jk}}{\partial x_j}-e_{jk}\frac{\partial S_{ik}}{\partial x_j}=\rho\frac{\partial^2 u_i}{\partial t^2} \quad (1)$$

$$\Delta f_i=\left(s_{ij}+S_{jk}w_{ik}+S_{ij}e-S_{ik}e_{jk}\right)n_j \quad (2)$$

where the usual convention for summation over repeated indices is applied.

The stress-strain relations are:

$$\left.\begin{aligned}S_{11}-S_{22}&=\mu_0\left(\lambda_1^2-\lambda_2^2\right)\\ S_{22}-S_{33}&=\mu_0\left(\lambda_2^2-\lambda_3^2\right)\\ S_{33}-S_{11}&=\mu_0\left(\lambda_3^2-\lambda_1^2\right)\end{aligned}\right\} \quad (3)$$

The cylindrical polar coordinated (r,θ,z) of a point in the initially deformed body are connected with rectangular coordinated by relations:

$$r=\sqrt{x_1^2+x_2^2},\quad \theta=\tan^{-1}(x_2/x_1),\quad z=x_3 \quad (4)$$

It is assumed that the only non-zero components of initial stress are S_{rr}, $S_{\theta\theta}$ and S_{zz} which are uniform throughout the body and the body is in the state of symmetrical incremental strain wrt z-axis. The equations of motion [Eqn (1)] reduce, in the cylindrical polar coordinates, to:

$$\begin{aligned}&\frac{\partial s_{rr}}{\partial r}+\frac{s_{rr}-s_{\theta\theta}}{\partial r}+\frac{\partial s_{rz}}{\partial z}-\left(S_{rr}-S_{zz}\right)\frac{\partial w_{rz}}{\partial z}=\rho\frac{\partial^2 u_r}{dt^2}\\ &\frac{\partial s_{zz}}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\,s_{zr}\right)-\left(S_{rr}-S_{zz}\right)\frac{1}{r}\frac{\partial}{\partial r}\left(r\,w_{rz}\right)=\rho\frac{\partial^2 u_z}{\partial t^2}\end{aligned} \quad (5)$$

The incremental displacements u_r and u_z in terms of potential function $\phi(r,z)$ are given by:

$$u_r=-\frac{\partial^2\phi}{\partial r\partial z},\quad u_z=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) \quad (6)$$

The function ϕ is given by the simple partial differential equation as

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2}\right\}\left\{K^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{\partial^2\phi}{\partial z^2}-\frac{\rho}{\mu_0}\frac{\partial^2\phi}{\partial t^2}\right\}=0 \quad (7)$$

where $K=\lambda_r/\lambda_z$. (8)

According to Kurashige[7], the components of incremental displacement and stress, in a steadystate, are given by:

$$u_r=\int_0^\infty\frac{\partial\bar{\phi}}{\partial z}\xi^2 J_1(r\xi)d\xi,\quad u_z=-\int_0^\infty\bar{\phi}\xi^3 J_0(r\xi)d\xi \quad (9)$$

$$\left.\begin{aligned} s_{zz} &= s - \mu_0\lambda_z^2\left(2+K^2\right)\int_0^\infty \xi^3 \frac{\partial\bar\phi}{\partial z} J_0(r\xi)d\xi \\ s_{zr} &= \frac{1}{2}\mu_0\lambda_z^2\left(1+K^2\right)\int_0^\infty\left(\xi^4\bar\phi + \xi^2\frac{\partial^2\bar\phi}{\partial z^2}\right)J_1(r\xi)d\xi \\ s_{rr} &= \mu_0\lambda_z^2\left\{\int_0^\infty\left(K^2\xi^2\frac{\partial\bar\phi}{\partial z}+\xi\frac{\partial^3\bar\phi}{\partial z^3}\right)J_0(r\xi)d\xi \right.\\ &\qquad \left. -\frac{2K^2}{r}\int_0^\infty \xi^2\frac{\partial\bar\phi}{\partial z}J_1(r\xi)d\xi\right\} \end{aligned}\right\} \quad (10)$$

where

$$s = \frac{1}{2}(s_{rr}+s_{00}) = \mu_0\lambda_z^2\int_0^\infty \xi\frac{\partial^3\bar\phi}{\partial z^3}J_0(r\xi)d\xi \quad (11)$$

and $\bar\phi$ is the Hankel transform of ϕ defined by

$$\left.\begin{aligned} \bar\phi &= \int_0^\infty \phi(r,z)\, rJ_0(r\xi)\, dr \\ \phi &= \int_0^\infty \bar\phi(r,z)\, \xi J_0(r\xi)\, d\xi \end{aligned}\right\} \quad (12)$$

where J_0 and J_1 are Bessel functions of order 0 and 1, respectively.

Equation (7), by Hankel's transform, reduces to the following ordinary differential equation, giving $\phi(r,z)$ for a steadystate:

$$\left(\frac{d^2}{dz^2}-\xi^2\right)\left(\frac{d^2\bar\phi}{dz^2}-K^2\xi^2\bar\phi\right)=0 \quad (13)$$

3. PUNCH PROBLEM

It is assumed that the semi-infinite medium $z \geq 0$ is initially deformed and the components S_{zz}, in addition to $S_{\theta\theta}$, is also zero so that

$$S_{rr} = \mu_0\left(\lambda_r^2 - \lambda_z^2\right) = -P \quad (14)$$

From Eqns (8) and (14), it is obvious that for no initial stress, when $P = 0$, $K = 1$.

The solution of the differential Eqn (13) is given by

$$\bar\phi = A(\xi)e^{-\xi z} + B(\xi)e^{-K\xi z} \quad (15)$$

The rigid punch is in the form of a solid of revolution which has the equation $z = f(r)$, referred to the tip of the punch as origin and it has a radius of contact 'a' with the medium (Fig. 1). If the pressure $p(r)$ is assumed to be applied in the plane $z = 0$, and the contact is free from friction, the boundary conditions are:

$$\left.\begin{aligned} u_r(r,0) &= p(r), \qquad (0 \leq r \leq \infty) \\ s_{rz} &= 0, \qquad (0 \leq r \leq \infty) \end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned} u_z &= D - f(r), \qquad (0 \leq r \leq a) \\ S_{zz} &= 0, \qquad (r > a) \end{aligned}\right\} \quad (17)$$

where D is a parameter whose physical significance is that it is the depth to which the tip of the punch penetrates the elastic half-space and $f(0) = 0$.

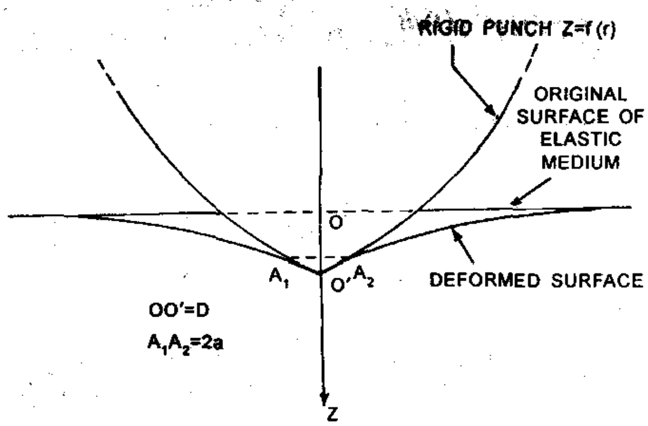

Figure 1. Indentation of a semi-infinite solid by an axi-symmetrical rigid punch.

Applying boundary conditions [Eqn (16)] to the solution [Eqn (15)], one has:

$$\left.\begin{aligned} A(\xi) &= \frac{1+K^2}{1-K^2}\,\frac{\bar P(\xi)}{\xi^2} \\ B(\xi) &= \frac{-2}{1-K^2}\,\frac{\bar P(\xi)}{\xi^2} \end{aligned}\right\} \quad (18)$$

The boundary conditions [Eqn (17)] give the following dual integral equations:

$$\int_0^\infty \xi \bar{p}(\xi) J_0(r\xi) d\xi = D - f(r), \quad (0 \le r \le a) \quad (19)$$

$$\int_0^\infty \xi^2 \bar{p}(\xi) J_0(r\xi) d\xi = 0, \quad (r > a) \quad (20)$$

Taking $\xi \bar{p}(\xi) = \Psi(a\xi)$, one has from Eqns (19) and (20)

$$\int_0^\infty \Psi(\zeta) J_0(x\zeta) d\zeta = D_1 - f_1(x), \quad (0 \le x \le 1) \quad (21)$$

$$\int_0^\infty \zeta\, \Psi(\zeta) J_0(x\zeta) d\zeta = 0, \quad (x \ge 1) \quad (22)$$

in which $D_1 = aD$, $f_1(x) = af(ax)$ and $a\xi = \zeta$

The solution of Eqn. (22) is given[13] as follows:

$$\Psi(\zeta) = \sqrt{(2/\pi)} \int_0^1 g(t) \cos(\zeta t) dt \quad (23)$$

Equation (21) is equivalent to the Abel's integral equation:

$$\sqrt{(2/\pi)} \int_0^r \frac{g(t)dt}{\sqrt{r^2 - t^2}} = D_1 - f_1(x), \quad (r > 0) \quad (24)$$

where the unknown functioin[13] $g(t)$ is given by

$$g(t) = \sqrt{(2/\pi)} \left\{ D_1 - t \int_0^t \frac{f_1'(x)}{\sqrt{t^2 - x^2}} Dx \right\} \quad (25)$$

and

$$D_1 = \int_0^1 \frac{f_1'(x) dx}{\sqrt{1 - x^2}} \quad (26)$$

or

$$D = a \int_0^a \frac{f(r)}{\sqrt{a^2 - r^2}} dr \quad (27)$$

From Eqn (23), $\psi(\xi)$ and hence $\psi(a\xi)$ is known. Thus $\bar{P}(\xi)$ being known, the components of stress and strain can be found.

4. SPECIAL CASES

4.1 Flat-Ended Circular Cylindrical Punch

Let the case be considered in which the semi-infinite elastic medium is deformed by the normal indentation of the boundary by a flat-ended circular cylinder of radius, a. Sinch in this case, the profile of the punch is not smooth at $r = a$, one must regard D as one of the data of problem. In this case $f_1(x) = 0$ and from Eqn (25), one gets:

$$g(t) = \sqrt{(2/\pi)}\, aD \quad (28)$$

Therefore, one has:

$$\bar{p}(x) = \frac{2D}{\pi} \frac{\sin a\xi}{\xi^2} \quad (29)$$

Hence

$$\bar{\phi} = \frac{2D}{\pi(1-K^2)} \left\{ (1+K^2) e^{-\xi z} - 2e^{-K\xi z} \right\} \frac{\sin a\xi}{\xi^4} \quad (30)$$

Thus, the non-vanishing components of incremental displacement and stresses are in terms of the Hankel inversion, Sneddon[14], as follows:

$$u_z = \frac{-2D}{\pi(1-K^2)} \int_0^\infty \left\{ (1+K^2) e^{-\xi z} - 2e^{-K\xi z} \right\} \frac{\sin a\xi}{\xi} J_0(r\xi)\, d\xi \quad (31)$$

$$u_r = \frac{-2D}{\pi(1-K^2)} \int_0^\infty \left\{ (1+K^2) e^{-\xi z} - 2Ke^{-K\xi z} \right\} \frac{\sin a\xi}{\xi} J_1(r\xi)\, d\xi \quad (32)$$

$$s_{zz} = \frac{2D}{\pi(1-K^2)} \mu_0 \lambda_z^2 \int_0^\infty \left\{ (1+K^2)^2 e^{-\xi z} - 4K^2 e^{-K\xi z} \right\} \sin a\xi\, J_0(r\xi) d\xi \quad (33)$$

$$s_{rr} = \frac{2D}{\pi(1-K^2)} \mu_0 \lambda_z^2 \left[\int_0^\infty \left\{ (1+K^2) e^{-\xi z} - 4K^3 e^{-K\xi z} \right\} \sin a\xi\, J_0(r\xi)\, d\xi - \frac{2K^2}{r} \int_0^\infty \left\{ (1+K^2) e^{-\xi z} - 2Ke^{-K\xi z} \right\} \frac{\sin a}{\xi} J_1(r\xi) d\xi \right] \quad (34)$$

$$s_{zr} = \frac{2D}{\pi\left(1-K^2\right)}\mu_0\lambda_z^2\int_0^\infty\left\{\left(1+K^2\right)^2 e^{-\xi z} - e^{-K\xi z}\right\} \sin a\xi\, J_1(r\xi)d\xi \quad (35)$$

4.2 Conical Punch

Here the boundary of the semi-inifinite elastic medium is deformed by a conical punch whose axis is normal to the indented plane. It is assumed that the axis of the cone coincides with the z-axis and that the vertex points downwards into the interior of the medium. In this case one takes:

$$f(r) = r\tan\alpha$$

where the semi-vertical angle β $\{= (\pi/2)-\alpha\}$ of the conical punch is supposed to be large, so that the conical punch is not very much pointed.

Therefore from Eqn. (27), one has:

$$D = \frac{1}{2}\pi\varepsilon, \quad \text{where } \varepsilon = a\tan\alpha$$

From Eqn (25), one obtains the expression:

$$g(t) = \sqrt{(2/\pi)}\ aD\,(1-t) \quad (36)$$

Thus one gets:

$$\bar{p}(\xi) = \frac{2D}{\pi a}\frac{(1-\cos a\xi)}{\xi^3} \quad (37)$$

Hence

$$\bar{\phi} = \frac{2D}{\pi a\left(1+K^2\right)}\left\{\left(1+K^2\right)e^{-\xi z} - 2e^{-K\xi z}\right\}\frac{\left(1-\cos a\xi\right)}{\xi^3} \quad (38)$$

Hence, the non-vanishing components of incremental displacement and stresses are in terms of Hankel's transformation as follows:

$$u_z = \frac{-2D}{\pi a\left(1-K^2\right)}\int_0^\infty\left\{\left(1+K^2\right)e^{-\xi z} - 2e^{-K\xi z}\right\}\frac{\left(1-\cos a\xi\right)}{\xi^2}J_0(r\xi)d\xi \quad (39)$$

$$u_r = \frac{-2D}{\pi a\left(1-K^2\right)}\int_0^\infty\left\{\left(1+K^2\right)e^{-\xi z} - 2Ke^{-K\xi z}\right\}\frac{\left(1-\cos a\xi\right)}{\xi^2}J_1(r\xi)d\xi \quad (40)$$

$$s_{zz} = \frac{2D}{\pi a\left(1-K^2\right)}\mu_0\lambda_z^2\left\{\int_0^\infty\left(1+K^2\right)^2 e^{-\xi z} - 4K^2e^{-K\xi z}\right\}\frac{\left(1-\cos a\xi\right)}{\xi^2}J_0(r\xi)d\xi \quad (41)$$

$$s_{rr} = \frac{2D}{\pi a\left(1-K^2\right)}\mu_0\lambda_z^2\left\{\int_0^\infty\left(1+K^2\right)^2 e^{-\xi z} - 4K^3e^{-K\xi z}\right\}\frac{\left(1-\cos a\xi\right)}{\xi^2}J_0(r\xi)d\xi$$
$$-\frac{2K^2}{r}\int_0^\infty\left\{\left(1+K^2\right)e^{-\xi z} - 4Ke^{-K\xi z}\right\}\frac{\left(1-\cos a\xi\right)}{\xi^2}J_1(r\xi)d\xi \quad (42)$$

$$s_{zr} = \frac{2D}{\pi a\left(1-K^2\right)}\mu_0\lambda_z^2\int_0^\infty\left\{\left(1+K^2\right)^2 e^{-\xi z} - e^{-K\xi z}\right\}\frac{\left(1-\cos a\xi\right)}{\xi}J_1(r\xi)d\xi \quad (43)$$

5. LIMITING CASE

The case of non-initial stress can be obtained by making $K\rightarrow 1$. The results so obtained agree with those already obtained by Sneddon[14] for materials which obey Hook's law. From the expressions of displacements and stresses, it appears that as $K\rightarrow 1$, all components of displacement and stresses tend to infinity which means that the situation becomes unstable.

6. NUMERICAL RESULTS & DISCUSSION

For a flat-ended circular cylindrical punch, variations of s_{zz}, s_{rr} and u_z with various parameters are shown in Figs. 2–5. As discussed earlier, for a non-initially

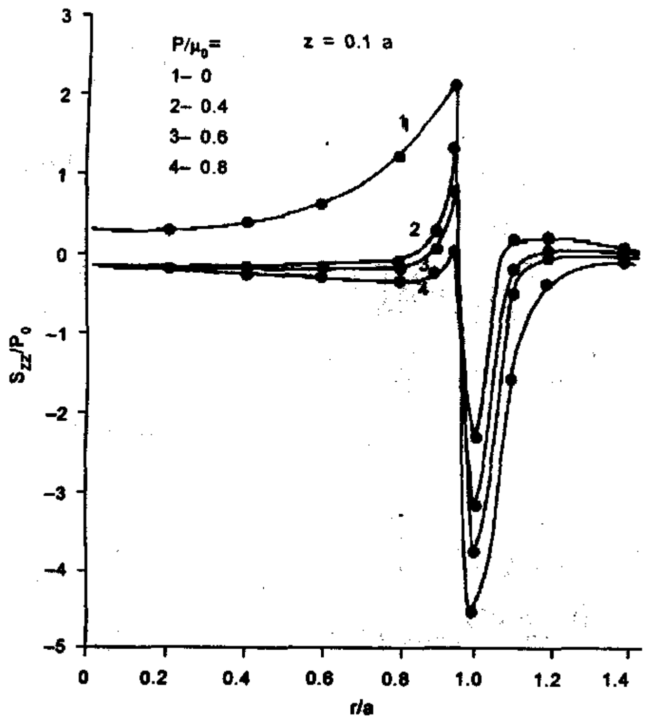

Figure 2. Variation of normal component of incremental stress s_{zz} with r.

stressed body, $P = 0$ which is given by K=1. For a body with high initial stress P/μ_0 tends to unity. Variations of stresses and displacement have been shown for values of P/μ_0 between 0 and 0.8, i.e. for cases between non-initially stressed state and those with high initial stress.

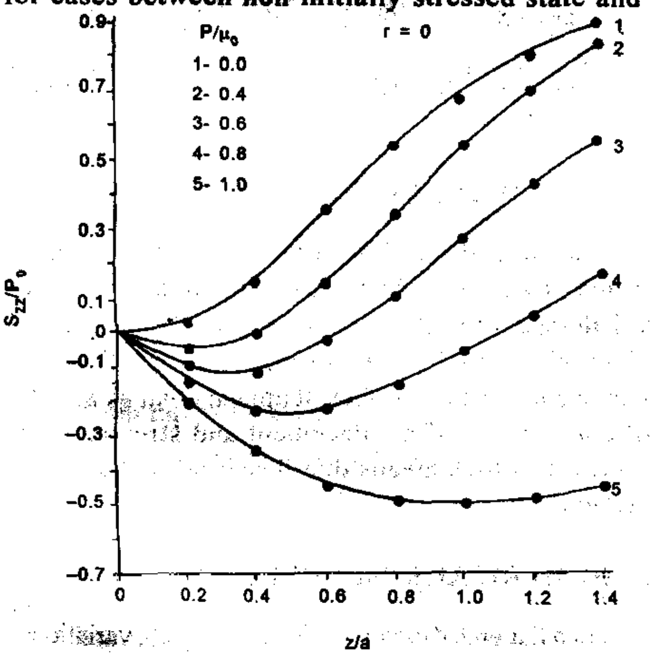

Figure 3. Variation of radial component of incremental stress s_{zz} with z.

(a) Figure 2 exhibits the variation of normal components of incremental stress s_{zz} with r at $z = 0.1a$. It shows that the lower the initial compressive stress is, the higher is the stress component under the punch. Near $r = a$, there is sharp rise and fall in the stress. In fact in the case of a flat-ended circular punch, the edge of the punch comes in contact with the elastic body which produces large stresses. This explains the discontinuity of stress near $r = a$.

(b) Figure 3 shows the variation of the radial component of incremental stress, s_{zz} with z along $r = 0$, i.e. z-axis. It is interesting to note that for the non-initially stressed body, s_{zz} is tensile and increases monotonically from zero. But for initially stressed bodies, it first decreases, and then increses monotonically, remaining compressive.

(c) Figure 4 gives the variation of incremental stress, s_{rr} with r. Higher the initial compressive stress is, the higher the incremental stress, s_{rr} is. It increases, monotonically from the centre of the punch and rises very sharply near the edge of the punch,

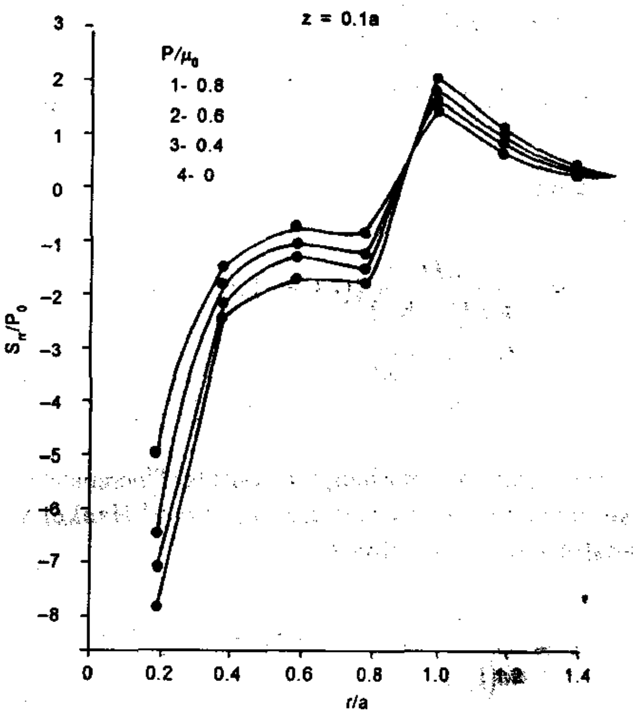

Figure 4. Variation of radial component of incremental stress s_{rr} with r.

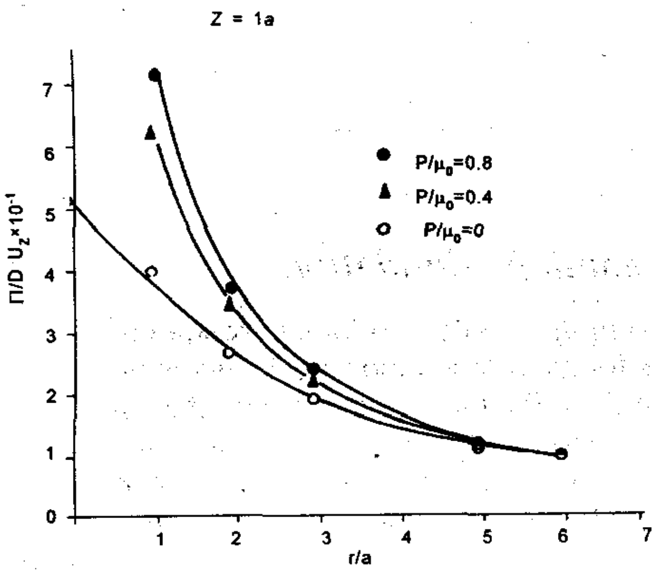

Figure 5. Variation of normal component of incremental displacement u_z with r.

given by $r = a$. After the edge, it continues to decrese gradually.

(d) Figure 5 shows the variation of the incremental displacement, u_z with r at $z = a$. It shows that higher the initial compressive force is, higher the incremental displacement u_r becomes.

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