Defence Science Journal, Vol. 53, No. 2, April 2003, pp. 157-165


# Unified Geometrically Nonlinear Formulation of All Higher-order Shear Deformation Theories for Cross-ply Plates

G.P. Dube, S. Kapuria and P.C. Dumir

*Indian Institute of Technology Delhi, New Delhi* – 110 016

**ABSTRACT**

Several higher-order shear deformation theories have been proposed for laminated plates, based on the expansions of displacements across the thickness, which are the same for all layers. In this study, a unified formulation of all higher-order theories is presented for cross-ply laminated plates based on polynomial expansions of displacements in the thickness coordinate $z$. It includes all the models available in literature. The governing equations for linear static and free-vibration response, and for buckling under inplane load are derived. The expressions for the stiffness matrix, inertia matrix, geometric stiffness matrix, and the load vector are developed for a simply supported rectangular plate using Navier's solution. A general purpose, single programme has been developed for all higher-order laminated plate theories.

**Keywords:** Plates, geometrical nonlinearity, cross-ply laminate, Navier's solution, higher-order shear deformation, higher-order laminated plate theories

## NOMENCLATURE

| | |
|---|---|
| $A$ | Transformation matrix |
| $A^k_{rs}$ | Generalised stiffness for the plate |
| $a, b$ | Plate sides |
| $h$ | Plate thickness |
| $h_f$ | Face sheet thickness |
| $L$ | Number of layers |
| $E_i$ | Young's modulus |
| $G_{ij}$ | Shear modulus |
| $\nu_{ij}$ | Poisson's ratio |
| $f^{mn}$ | Fourier coefficients of $f$ |
| $F_k, F_{ik}$ | Generalised force resultant and its elements |
| $I_k$ | Generalised inertia |
| $p_z$ | Generalised load |
| $k^2_{xz}, k^2_{yz}$ | Shear correction factors |
| $K, K'$ | Stiffness matrices |
| $K_G, K'_G$ | Geometrical stiffness matrices |
| $M, M'$ | Inertia matrices |
| $M_x, M_y, M_{xy}$ | Moments |
| $N_x, N_y, N_{xy}$ | Inplane forces |
| $P, P'$ | Load vectors |
| $Q_{rs}(i)$ | Stiffness terms in $\sigma$ - $\varepsilon$ relations |
| $\rho(i)$ | Density |

Received 24 October 2002

| | |
|---|---|
| $Q_x, Q_y$ | Shear forces |
| $u, v, w$ | Displacements |
| $u_i, v_i, w_j$ | Series terms in displacements |
| $x, y, z$ | Cartesian coordinates |
| $t$ | Time |
| $\alpha, \beta$ | $m\pi/a$, $n\pi/b$ |
| $\omega$ | Natural frequency |
| $\varepsilon_i$ | Strains |
| $\sigma_i$ | Stresses |

## 1. INTRODUCTION

For the efficient design of laminated composite and sandwich plates, a good understanding of their deformation characteristics under various load conditions are needed. Classical plate theory, first-order shear deformation theory (FSDT), and higher-order shear deformation theories (HSDTs) involving higher-order terms in the Taylor's expansion of the displacements in the thickness coordinate $z$ have been developed for orthotropic and laminated plates. Lo[1,2], *et al.* have presented, for a laminated plate, a closed-form solution with higher-order shear deformation theories, including the effect of transverse normal strain. Kant[3] derived the variationally consistent third-order theory for symmetrically laminated plate, including the distortion of the transverse normals and the effect of transverse normal stress/strain. Reddy[4,5] derived a third-order variationally consistent theory which satisfies the conditions of zero shear stress on the faces of the plate. Using the theory of Reddy, Senthilnathan[6], *et al.* presented a simplified HSDT by splitting up the transverse displacement into bending and shear contributions. Pandya and Kant[7], and Kant and Manjunatha[8] have presented third-order HSDTs including the transverse normal strain in the former, for laminated cross-ply and sandwich plates and have given corresponding finite-element formulations. Noor and Burton[9] presented an assessment of first-order shear deformation theories and HSDTs for the static, free vibration, and buckling analyses of laminated composite plates. Srinivas[10], *et al.* Srinivas and Rao[11], and Noor[12] presented exact three-dimensional elasticity solutions for the free vibration of isotropic, orthotropic, anc anisotropic composite laminated plates. Swaminathar and Kant[13,14] have recently compared five non-classica plate theories for deflections and stresses unde transverse loads, natural frequencies of free vibrations and buckling loads under inplane static loads, fo cross-ply composite and sandwich simply-supporte plates. Pagano[15], and Pagano and Hatfield[16] hav given exact solutions for the rectangular composit and sandwich plates. Noor[17] has given elasticit solutions for stability of multilayered composit plates.

The objective of this study is to present a unifie general formulation of all higher-order theories fo geometrically nonlinear responses of cross-pl plates, based on a single polynomial expansion o displacements in the thickness coordinate $z$. It include ten models studied by Swaminathan[13] as specia cases. The governing equations for linear stati response under transverse load, free-vibration respons and for buckling under inplane load, have bee derived. The expressions for the stiffness matri $K$, inertia matrix $M$, geometric stiffness matrix $K_G$ and the load vector $P$ have been developed for simply supported rectangular plate using Navier' solution. A general purpose, single programme ha been developed for all higher-order laminated plat theories.

## 2. UNIFIED FORMULATION OF GOVERNING EQUATIONS

Consider a laminated cross-ply composite c sandwich plate of sides $a$, $b$ along axes $x$, $y$ an thickness $h$ with its mid-plane at $z = 0$. Summatio convention is used with the summation indice $i$, $i'$ ranging from 0 to $p$; $j$, $j'$ ranging from 0 t $q$; and $r$, $s$ ranging from 1 to 6. The displacemen are expanded as polynomials in the thicknes coordinate $z$:

$$\begin{aligned} u(x,y,z,t) &= z^i u_i(x,y,t) \\ v(x,y,z,t) &= z^i v_i(x,y,t) \\ w(x,y,z,t) &= z^j w_j(x,y,t) \end{aligned} \quad (1$$

The number of terms $p+1$ in inplane displacemen can be different from the number of terms $q+1$ i the transverse displacements. The virtual displacements are given by

$$\begin{aligned} \delta u(x,y,z,t) &= z^i \delta u_i(x,y,t) \\ \delta v(x,y,z,t) &= z^i \delta v_i(x,y,t) \\ \delta w(x,y,z,t) &= z^j \delta w_j(x,y,t) \end{aligned} \qquad (2)$$

In the strain-displacement relations, the nonlinearity is included only in the inplane strains due to $w$ alone

$$\begin{aligned} \varepsilon_1 &= \varepsilon_x = u_{,x} + \tfrac{1}{2} w_{,x}^2 \\ \varepsilon_2 &= \varepsilon_y = v_{,y} + \tfrac{1}{2} w_{,y}^2 \\ \varepsilon_3 &= \varepsilon_z = w_{,z} \\ \varepsilon_4 &= \gamma_{yz} = v_{,z} + w_{,y} \\ \varepsilon_5 &= \gamma_{zx} = u_{,z} + w_{,x} \\ \varepsilon_6 &= \gamma_{xy} = u_{,y} + v_{,x} + w_{,x} w_{,y} \end{aligned} \qquad (3)$$

where subscript comma denotes partial differentiation. The strain increments $\delta\varepsilon_i$ for $\delta u$, $\delta v$, $\delta w$ are:

$$\begin{aligned} \delta\varepsilon_1 &= z^i \delta u_{i,x} + z^{j+j'} w_{j',x} \delta w_{j,x} \\ \delta\varepsilon_2 &= z^i \delta v_{i,y} + z^{j+j'} w_{j',y} \delta w_{j,y} \\ \delta\varepsilon_3 &= j z^{j-1} \delta w_j \\ \delta\varepsilon_4 &= i z^{i-1} \delta v_i + z^j \delta w_{j,y} \\ \delta\varepsilon_5 &= i z^{i-1} \delta u_i + z^j \delta w_{j,x} \\ \delta\varepsilon_6 &= z^i (\delta u_{i,y} + \delta v_{i,x}) + z^{j+j'} \\ &\quad (w_{j',y} \delta w_{j,x} + w_{j',x} \delta w_{j,y}) \end{aligned} \qquad (4)$$

Two models of linear elastic constitutive equations are used.

(a) If $\varepsilon_3 \neq 0$, i.e., $q \geq 1$, then actual Young's moduli are used for orthotropic material.

(b) If $\varepsilon_3 = 0$, i.e., $q = 0$, then reduced moduli based on the approximation, $\sigma_z = 0$ are used.

The constitutive equations are:

$$\begin{aligned} \sigma_1 &= \sigma_x = Q_{11}\varepsilon_1 + Q_{12}\varepsilon_2 + Q_{13}\varepsilon_3 \\ \sigma_2 &= \sigma_y = Q_{12}\varepsilon_1 + Q_{22}\varepsilon_2 + Q_{23}\varepsilon_3 \\ \sigma_3 &= \sigma_z = Q_{13}\varepsilon_1 + Q_{23}\varepsilon_2 + Q_{33}\varepsilon_3 \\ \sigma_4 &= \tau_{yz} = Q_{44}\gamma_{yz} \\ \sigma_5 &= \tau_{zx} = Q_{55}\gamma_{zx} \\ \sigma_6 &= \tau_{xy} = Q_{66}\gamma_{xy} \end{aligned} \qquad (5)$$

where

$$Q_{44} = G_{yz}, \quad Q_{55} = G_{xz}, \; Q_{66} = G_{xy}.$$

*For Case* (a)

$$\begin{aligned} Q_{11} &= E_x(1 - \nu_{yz}\nu_{zy})/\Delta \\ Q_{12} &= E_x(\nu_{yx} + \nu_{zx}\nu_{yz})/\Delta \\ Q_{13} &= E_x(\nu_{zx} + \nu_{yx}\nu_{zy})/\Delta, \\ Q_{22} &= E_y(1 - \nu_{xz}\nu_{zx})/\Delta \\ Q_{23} &= E_y(\nu_{zy} + \nu_{xy}\nu_{zx})/\Delta \\ Q_{33} &= E_z(1 - \nu_{xy}\nu_{yx})/\Delta \\ \Delta &= 1 - \nu_{xy}\nu_{yx} - \nu_{yz}\nu_{zy} - \nu_{zx}\nu_{xz} - 2\nu_{xy}\nu_{yz}\nu_{zx} \end{aligned} \qquad (6)$$

*For Case* (b)

$$\begin{aligned} Q_{11} &= E_x/(1 - \nu_{xy}\nu_{yx}) \\ Q_{22} &= E_y/(1 - \nu_{xy}\nu_{yx}) \\ Q_{12} &= \nu_{yx}E_x/(1 - \nu_{xy}\nu_{yx}) \\ Q_{13} &= Q_{23} = Q_{33} = 0 \end{aligned} \qquad (7)$$

The 6 × 1 generalised force resultant matrices $F_k$ for the mid-plane are defined as the integral of the product of 6 × 1 stress matrix $\sigma$ and the $k^{th}$ power of $z$ across the thickness:

$$\begin{aligned} \sigma &= [\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6]^T \\ F_k &= [F_{1k} \; F_{2k} \; F_{3k} \; F_{4k} \; F_{5k} \; F_{6k}]^T \\ &= \int_{-h/2}^{h/2} \sigma z^k dz, \quad k = 0,1,2,\ldots \end{aligned} \qquad (8)$$

The inplane forces $N_x$, $N_y$, $N_{xy}$, transverse shear forces $Q_x$, $Q_y$, and moments $M_x$, $M_y$, $M_{xy}$ are related to the elements $F_{ik}$ of $F_k$ as $N_x = F_{10}$, $N_y = F_{20}$, $N_{xy} = F_{60}$, $Q_x = F_{50}$, $Q_y = F_{40}$, $M_x = F_{11}$, $M_x = F_{11}$, $M_y = F_{21}$, $M_{xy} = F_{61}$. The generalised inertia $I_k$ and the generalised transverse load $p_{z_k}$ for the mid-plane are defined as

$$I_k = \int_{-h/2}^{h/2} \rho z^k dz \qquad p_{z_k} = (\sigma_z z^k)\Big|_{-h/2}^{h/2} \tag{9}$$

$$k = 0,1,2\ldots$$

The following equations of motion and boundary conditions are obtained using Hamilton's principle:

$$\begin{aligned} &I_{i+i'}\ddot{u}_{i'} - F_{1i,x} - F_{6i,y} + iF_{5(i-1)} = 0 \\ &I_{i+i'}\ddot{v}_{i'} - F_{6i,x} - F_{2i,y} + iF_{4(i-1)} = 0 \\ &\qquad i = 0,\ldots,p \end{aligned} \tag{10}$$

$$\begin{aligned} &I_{j+j'}\ddot{w}_{j'} - F_{5j,x} - F_{4j,y} + jF_{3(j-1)} \\ &-[F_{1(j+j')}w_{j',x}]_{,x} - [F_{2(j+j')}w_{j',y}]_{,y} \\ &-[F_{6(j+j')}w_{j',y}]_{,x} - [F_{6(j+j')}w_{j',x}]_{,y} = p_{z_j} \\ &\qquad j = 0,\ldots,q \end{aligned} \tag{11}$$

at $x = 0$, $a$: prescribed values of

$F_{1i}$ or $u_i$, $F_{6i}$ or $v_i$

$$F_{1(j+j')wj',x} + F_{6(j+j')wj',y} + F_{5j} \text{ or } w_j \tag{12}$$

at $y = 0$, b: prescribed values of

$F_{6i}$ or $u_i$, $F_{2i}$ or $v_i$,

$$F_{2(j+j')}w_{j',y} + F_{6(j+j')}w_{j',x} + F_{4j} \text{ or } w_j \tag{13}$$

Neglecting the nonlinear terms in Eqns (10) to (12), yields the linear equations of motion:

$$\begin{aligned} &I_{i+i'}\ddot{u}_{i'} - F_{1i,x} - F_{6i,y} + iF_{5(i-1)} = 0, \\ &\qquad i = 0,\ldots,p \\ &I_{i+i'}\ddot{v}_{i'} - F_{6i,x} - F_{2i,y} + iF_{4(i-1)} = 0, \\ &\qquad i = 0,\ldots,p \\ &I_{j+j'}\ddot{w}_{j'} - F_{5j,x} - F_{4j,y} + jF_{3(j-1)} = p_{z_j} \\ &\qquad j = 0,\ldots,q. \end{aligned} \tag{14}$$

and the linear boundary conditions:

at $x = 0$, $a$: prescribed values of $F_{1i}$ or $u_i$,

$$F_{6i} \text{ or } v_i, \quad F_{5j} \text{ or } w_j \tag{15}$$

at $y = 0$, $b$: prescribed values of $F_{6i}$ or $u_i$,

$$F_{2i} \text{ or } v_i, \quad F_{4j} \text{ or } w_j \tag{16}$$

The linear strain-displacement relations are obtained from Eqn (3):

$$\begin{aligned} &\varepsilon_1 = z^i u_{i,x}, && \varepsilon_2 = z^i v_{i,y}, \\ &\varepsilon_3 = jz^{j-1}w_j && \varepsilon_4 = iz^{i-1}v_i + z^j w_{j,y} \\ &\varepsilon_5 = iz^{i-1}u_i + z^j w_{j,x} && \varepsilon_6 = z^i(u_{i,y} + v_{i,x}) \end{aligned} \tag{17}$$

Equations (5), (8), (17) yield following relations for the generalised force resultants for the linear case:

$$\begin{aligned} F_{1k} &= A_{11}^{k+i}u_{i,x} + A_{12}^{k+i}v_{i,y} + jA_{13}^{k+j-1}w_j \\ F_{2k} &= A_{12}^{k+i}u_{i,x} + A_{22}^{k+i}v_{i,y} + jA_{23}^{k+j-1}w_j \\ F_{3k} &= A_{13}^{k+i}u_{i,x} + A_{23}^{k+i}v_{i,y} + jA_{33}^{k+j-1}w_j \\ F_{4k} &= iA_{44}^{k+i-1}v_i + A_{44}^{k+j}w_{j,y} \\ F_{5k} &= iA_{55}^{k+i-1}u_i + A_{55}^{k+j}w_{j,x} \\ F_{6k} &= A_{66}^{k+i}(u_{i,y} + v_{i,x}) \end{aligned} \tag{18}$$

where

$A_{rs}^k = \int_{-h/2}^{h/2} Q_{rs} z^k dz$ are the generalised stiffness of the plate.

$$A_{rs}^0 = A_{rs}, A_{rs}^1 = B_{rs}, A_{rs}^2 = D_{rs}$$

where $A$, $B$, $D$ are the inplane, coupling and bending stiffness of the plate, respectively. For first-order shear deformation theroies:

$$A_{44}^0 = k_{sy}^2 \int_{-h/2}^{h/2} Q_{44}dz, \quad A_{55}^0 = k_{sy}^2 \int_{-h/2}^{h/2} Q_{55}dz \tag{19}$$

where $k_{sx}^2, k_{sy}^2$ are the shear correction factors for shears $Q_x$, $Q_y$, respectively.

The governing equations for buckling under inplane load for symmetric laminate are obtained as follows. The pre-buckling linear solution consists of constant values of $F_{1i}$, $F_{2i}$, $F_{6i}$ which satisfy

Eqn (10) with $F_{4(i-1)} = 0$ $F_{5(i-1)} = 0$. Using this in Eqn (11) yields for $j = 0, \ldots, q$:

$$-F_{5j,x} - F_{4j,y} + jF_{3(j-1)} - F_{1(j+j')}w_{j',xx} \\ -F_{2(j+j')}w_{j',yy} - 2F_{6(j+j')}w_{j',xy} = 0. \tag{20}$$

## 3. NAVIER'S SOLUTION

The displacement equations for linear dynamic response are obtained by using $F_{ik}$ from Eqn (18) in Eqn (14):

$$\begin{aligned} &I_{i+i'}\ddot{u}_{i'} - A_{11}^{i+i'}u_{i',xx} - A_{12}^{i+i'}v_{i',xy} \\ &- j'A_{13}^{i+j'-1}w_{j',x} - A_{66}^{i+i'}(u_{i',yy} + v_{i',xy}) \\ &+ i(i'A_{55}^{i+i'-2}u_{i'} + A_{55}^{i+j'-1}w_{j',x}) = 0 \\ &I_{i+i'}\ddot{v}_{i'} - A_{66}^{i+i'}(u_{i',xy} + v_{i',xx}) - A_{12}^{i+i'}u_{i',xy} \\ &- A_{22}^{i+i'}v_{i',yy} - j'A_{23}^{i+j'-1}w_{j',y} \\ &+ i(i'A_{44}^{i+i'-2}v_{i'} + A_{44}^{i+j'-1}w_{j',y}) = 0 \\ &I_{j+j'}\ddot{w}_{j'} - i'A_{55}^{j+i'-1}u_{i',x} - A_{55}^{j+j'}w_{j',xx} \\ &- i'A_{44}^{j+i'-1}v_{i',y} - A_{44}^{j+j'}w_{j',yy} + j \\ &[A_{13}^{j+i'-1}u_{i',x} + A_{23}^{j+i'-1}v_{i',y} + j'A_{33}^{j+j'-2}w_{j'}] = p_{z_j} \end{aligned} \tag{21}$$

for $i = 0,\ldots,p$; and $j = 0,\ldots q$. The boundary conditions for simply-supported plate are taken as

At $x = 0$, $a$: $F_{1i} = 0$, $v_i = 0$, $w_j = 0$;

at $y = 0$, $b$: $\quad F_{2i} = 0$, $u_i = 0$, $w_j = 0$ (22)

The $u_i$, $v_i$, $w_j$ are expanded in the following series form which satisfy all conditions [Eqn (22)]:

$$\begin{aligned} u_i &= \sum\sum u_i^{mn}\cos\alpha x\sin\beta y\cos\omega t \\ w_j &= \sum\sum w_j^{mn}\sin\alpha x\sin\beta y\cos\omega t \\ v_i &= \sum\sum v_i^{mn}\sin\alpha x\cos\beta y\cos\omega t \\ p_{z_j} &= \sum\sum p_{z_j}^{mn}\sin\alpha x\sin\beta y\cos\omega t \end{aligned} \tag{23}$$

where $\alpha = m\pi/a$, $\beta = n\pi/b$ and $\omega$ is the frequency. Substituting $u_i$, $v_i$, $w_j$, $p_{z_j}$ from Eqn (23) in Eqn (21) yields:

$$-\omega^2\mathrm{M}\mathrm{u}^* + \mathrm{K}\mathrm{u}^* = \mathrm{P} \tag{24}$$

where $u^* = [u_0\ v_0\ w_0\ u_1\ v_1\ w_1\ u_2\ v_2\ w_2\ \ldots]^T$, $M$ is the inertia matrix, $K$ is the stiffness, and $P$ is the load vector. The non-zero elements of matrices $M$, $K$ and $P$ are:

$$\begin{aligned} &M(3i+1,3i'+1) = I_{i+i'} \\ &M(3i+2,3i'+2) = I_{i+i'} \\ &M(3j+3,3j'+3) = I_{j+j'} \\ &K(3i+1,3i'+1) = \alpha^2A_{11}^{i+i'} + \beta^2A_{66}^{i+i'} + ii'A_{55}^{i+i'-2} \\ &K(3i+1,3i'+2) = \alpha\beta(A_{12}^{i+i'} + A_{66}^{i+i'}) \\ &K(3i+1,3j'+3) = \alpha iA_{55}^{i+j'-1} - \alpha j'A_{13}^{i+j'-1} \\ &K(3i+2,3i'+1) = \alpha\beta(A_{12}^{i+i'} + A_{66}^{i+i'}) \\ &K(3i+2,3i'+2) = \alpha^2A_{66}^{i+i'} + \beta^2A_{22}^{i+i'} + ii'A_{44}^{i+i'-2} \\ &K(3i+2,3j'+3) = \beta iA_{44}^{i+j'-1} - \beta j'A_{23}^{i+j'-1} \\ &K(3j+3,3i'+1) = \alpha i'A_{55}^{j+i'-1} - \alpha jA_{13}^{j+i'-1} \\ &K(3j+3,3i'+2) = \beta i'A_{44}^{j+i'-1} - \beta jA_{23}^{j+i'-1} \\ &K(3j+3,3j'+3) = \alpha^2A_{55}^{j+j'} + \beta^2A_{44}^{j+j'} + jj'A_{33}^{j+j'-2} \\ &P(3j+3) = p_{z_j}^{mn} \end{aligned} \tag{25}$$

for $i = 0,\ldots, p$, $i' = 0,\ldots, p$, $j = 0,\ldots, q$, $j' = 0,\ldots, q$.

For the buckling problem under inplane loads, let the loads be increased proportionately with $F_{1k} = \lambda\bar{F}_{1k}$, $F_{2k} = \lambda\bar{F}_{2k}$, $F_{6k} = \lambda\bar{F}_{6k}$, where $\bar{F}_{1k}$ $\bar{F}_{2k}$ $\bar{F}_{6k}$ define the proportion of loads and $\lambda$ is the buckling parameter. The Navier's solution of Eqn (20) is obtained by substituting from Eqn (23) and setting $u = v = 0$:

$$\mathrm{K}\mathrm{u}^* - \lambda\mathrm{K}_G\mathrm{u}^* = 0 \tag{26}$$

with non-zero elements of the geometric stiffness matrix $K_G$ being:

$$\begin{aligned} &K_G(3j+3,3j'+3) \\ &= -\alpha^2\bar{F}_{1(j+j')} - \beta^2\bar{F}_{2(j+j')} + 2\alpha\beta\bar{F}_{6(j+j')} \end{aligned} \tag{27}$$

One-term static solution for a simply supported plate subjected to a sinusoidal load on the top surface, i.e.,

$$\sigma_z(x, y, \tfrac{h}{2}) = p_0\sin\alpha x\sin\beta y \Rightarrow p_{z_j}^{mn} = p_0(\tfrac{h}{2})^j$$

is obtained by solving $Ku^* = P$ for $u^*$. The force resultants are computed using Eqn (18). The displacements, strains and stresses at point $z$ in layer number $il$ are obtained using Eqns (1), (3) (retaining only linear terms) and (5).

For FSDT models, a better estimate of $\tau_{xz}$, $\tau_{YZ}$, $\sigma_z$ is obtained by integrating the equations of equilibrium across the thickness. The equilibrium equations for x and y directions are integrated to yield $\tau_{xz}(z)$, $\tau_{yz}(z)$, respectively. The equilibrium equation for $z$ direction is then integrated to yield $\sigma_z(z)$.

$$
\begin{aligned}
&\sigma_{x,x}+\tau_{yx,y}+\tau_{zx,z}=0 \;\Rightarrow\\
&\tau_{zx}^{mn}(z)=\sigma_5^{mn}(z)=\int_0^z(-\alpha\sigma_1^{mn}+\beta\sigma_6^{mn})dz\\
&\tau_{xy,x}+\sigma_{y,y}+\tau_{zy,z}=0 \;\Rightarrow\\
&\tau_{zy}^{mn}(z)=\sigma_4^{mn}(z)=\int_0^z(-\beta\sigma_2^{mn}+\alpha\sigma_6^{mn})dz\\
&\tau_{zx,x}+\tau_{yz,y}+\sigma_{z,z}=0 \;\Rightarrow\\
&\sigma_z^{mn}(z)=\sigma_3^{mn}(z)=\int_0^z(\alpha\sigma_5^{mn}+\beta\sigma_4^{mn})dz
\end{aligned}
\tag{28}
$$

since $\tau_{zx}^{mn}(-h/2)=\tau_{zy}^{mn}(-h/2)=\sigma_z^{mn}(-h/2)=0$ at the traction free bottom face. Equation (28) is integrated layerwise.

To obtain frequencies of free oscillations of the $(m,n)^{th}$ mode, equations $Ku^* = w^2Mu^*$ are solved for all eigenvalues.

The buckling problem is considered for the following prescribed inplane loads at the boundary:

$$\bar{N}_x=\bar{F}_{10},\ \bar{N}_y=\bar{F}_{20},\ \bar{N}_{xy}=\bar{F}_{60}$$
$$i.e.\ \bar{N}_x:\bar{N}_y:\bar{N}_{xy}::\bar{F}_{10}:\bar{F}_{20}:\bar{F}_{60}$$

and

$$\bar{F}_{1i}=\bar{F}_{2i}=\bar{F}_{6i}=0 \quad \text{for} \quad i\neq 0$$

For given $(m,n)$ $Ku^* = \lambda K_G u^*$ is solved for the smallest eigenvalue $\lambda$ and reworked for other values of $(m,n)$ to obtain the absolutely smallest value of $\lambda$.

A computer program has been developed for solving static response, linear vibration frequencies, and buckling loads for any cross-ply composite/sandwich, simply supported rectangular plate using the general unified formulation presented herein for any higher-order plate theory using single displacement expansion across the thickness.

**Table 1. Identification of elements of displacement vector $u^*$ and variables used in various theories**

| $z^i$ | $i^{th}$ element of $u^*$ | Variable | Models | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 6 | 2 | 7 | 3 | 8 | 4 | 9 | 5 | 10 |
| $z^0$ | 1 | $u_0$ | | $u_0$ | | $u_0$ | | $u_0$ | | $u_0$ | | $u_0$ |
| $z^0$ | 2 | $v_0$ | | $v_0$ | | $v_0$ | | $v_0$ | | $v_0$ | | $v_0$ |
| $z^0$ | 3 | $w_0$ | $w_0$ | $w_0$ | $w_0$ | $w_0$ | $w_0$ | $w_0$ | $w_0^b+w_0^s$ | $w_0^b+w_0^s$ | $w_0$ | $w_0$ |
| $z$ | 4 | $u_1$ | $\theta_x$ | $\theta_x$ | $\theta_x$ | $\theta_x$ | $\theta_x$ | $\theta_x$ | $-w_{0,x}^b$ | $-w_{0,x}^b$ | $\theta_x$ | $\theta_x$ |
| $z$ | 5 | $v_1$ | $\theta_y$ | $\theta_y$ | $\theta_y$ | $\theta_y$ | $\theta_y$ | $\theta_y$ | $-w_{0,y}^b$ | $-w_{0,y}^b$ | $\theta_y$ | $\theta_y$ |
| $z$ | 6 | $w_1$ | | $\theta_z$ | | | | | | | | |
| $z^2$ | 7 | $u_2$ | | $u_0^*$ | | $u_0^*$ | | | | | | |
| $z^2$ | 8 | $v_2$ | | $v_0^*$ | | $v_0^*$ | | | | | | |
| $z^2$ | 9 | $w_2$ | $w_0^*$ | $w_0^*$ | | | | | | | | |
| $z^3$ | 10 | $u_3$ | $\theta_x^*$ | $\theta_x^*$ | $\theta_x^*$ | $-\theta_x^*$ | $-\frac{4}{3h^2}(\theta_x+w_{0,x})$ | $-\frac{4}{3h^2}(\theta_x+w_{0,x})$ | $-\frac{4}{3h^2}w_{0,x}^s$ | $-\frac{4}{3h^2}w_{0,x}^s$ | | |
| $z^3$ | 11 | $v_3$ | $\theta_y^*$ | $\theta_y^*$ | $\theta_y^*$ | $-\theta_y^*$ | $-\frac{4}{3h^2}(\theta_y+w_{0,y})$ | $-\frac{4}{3h^2}(\theta_y+w_{0,y})$ | $-\frac{4}{3h^2}w_{0,x}^s$ | $-\frac{4}{3h^2}w_{0,x}^s$ | | |
| $z^3$ | 12 | $w_3$ | | $\theta_z^*$ | | | | | | | | |
| size | | | 6 | 12 | 5 | 9 | 5 | 7 | 5 | 7 | 3 | 5 |

## 4. EXISTING THEORIES AS PARTICULAR CASES OF UNIFIED FORMULATION

The ten theories (model 1 to model 10) studied by Swaminathan[13] are the particular cases of the unified formulation presented herein as shown in Table 1.

Model 1 to model 5 are for symmetric laminates and model 6 to model 10 are for unsymmetric laminates. Model 1 and model 6 have been originally presented by Kant[3], Pandya and Kant[7]; model 2 and model 7 by Pandya and Kant[7]; model 3 and model 8 by Reddy[4]; model 4 and model 9 by Senthilnathan[6], *et al.* and model 5 and model 10 are FSDT models with shear correction factors. $k_{sx}^2 = k_{sx}^2 = 5/6$ in models 3,8,4, and 9, the size of the assembled $M$, $K$, $K_G$, $P$ matrices is reduced from 5, 7, 5, 7 to 3, 5, 2, 4, respectively using the transformation matrix $A$ from the actual vector $u'^*$ formed by the independent displacement variables used in the formulation to the vector $u^*$ of the generalised formulation presented herein: $u^* = Au'^*$. Equations (24) and (26) can be unified as

$$\begin{aligned} &-\omega^2 M u^* + (K - K_G) u^* = P \\ &\Rightarrow \quad -\omega^2 M' u'^* + (K' - K_G') u'^* = P' \end{aligned} \tag{29}$$

where the reduced matrices $M'$, $K'$, $K'_G$, $P'$ are related to the matrices $M$, $K$, $K_G$, $P$ by

$$\begin{aligned} &M' = A^T M A, \quad K' = A^T K A \\ &K_G' = A^T K_G A, \quad P' = A^T P \end{aligned} \tag{30}$$

The transformation matrices A for models 3, 4, 8, and 9 are given respectively by

$$u^* = \begin{bmatrix} w_0^{mn} \\ u_1^{mn} \\ v_1^{mn} \\ u_3^{mn} \\ v_3^{mn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{4\alpha}{3h^2} & -\frac{4}{3h^2} & 0 \\ -\frac{4\beta}{3h^2} & 0 & -\frac{4}{3h^2} \end{bmatrix} \begin{bmatrix} w_0^{mn} \\ \theta_x^{mn} \\ \theta_y^{mn} \end{bmatrix}$$

$$u^* = \begin{bmatrix} w_0^{mn} \\ u_1^{mn} \\ v_1^{mn} \\ u_3^{mn} \\ v_3^{mn} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\alpha & 0 \\ -\beta & 0 \\ 0 & -\frac{4\alpha}{3h^2} \\ 0 & -\frac{4\beta}{3h^2} \end{bmatrix} \begin{bmatrix} w_0^{bmn} \\ w_0^{smn} \end{bmatrix}$$

$$u^* = \begin{bmatrix} u_0^{mn} \\ v_0^{mn} \\ w_0^{mn} \\ u_1^{mn} \\ v_1^{mn} \\ u_3^{mn} \\ v_3^{mn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{4\alpha}{3h^2} & -\frac{4}{3h^2} & 0 \\ 0 & 0 & -\frac{4\beta}{3h^2} & 0 & -\frac{4}{3h^2} \end{bmatrix} \begin{bmatrix} u_0^{mn} \\ v_0^{mn} \\ w_0^{mn} \\ \theta_x^{mn} \\ \theta_y^{mn} \end{bmatrix}$$

$$u^* = \begin{bmatrix} u_0^{mn} \\ v_0^{mn} \\ w_0^{mn} \\ u_1^{mn} \\ v_1^{mn} \\ u_3^{mn} \\ v_3^{mn} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\frac{4\alpha}{3h^2} \\ 0 & 0 & 0 & -\frac{4\beta}{3h^2} \end{bmatrix} \begin{bmatrix} u_0^{mn} \\ v_0^{mn} \\ w_0^{bmn} \\ w_0^{smn} \end{bmatrix} \tag{31}$$

After finding solution for $u'^*$ for models 3, 4, 8, and 9; use $u^* = Au'^*$ to obtain $u^*$.

The general shear correction factors $k_{sx}^2, k_{sy}^2$ are based on the quadratic variation of shear stress across the thickness:

$$\tau_{zx} \approx \tfrac{3}{2}\tau_m (1 - 4z^2/4h^2)$$

where $\tau_m$ is the mean stress.

The shear strain energy based on $\tau_m$ across the thickness is modified by the factor $k_{sx}^2$ so that $\int(\tau_m^2/2G_{55})dz/k_{sx}^2 = \int(\tau_{zx}^2/2G_{55})dz$

yielding

$$k_{sx}^2 = \frac{4\sum_{i=1}^{L}[z_u(i) - z_l(i)]/Q_{55}(i)}{9\sum_{i=1}^{L}\frac{\left[\begin{array}{l} z_u(i) - z_l(i) - \frac{8}{3h^2}\{z_u^3(i) - z_l^3(i)\} \\ + \frac{16}{5h^4}\{z_u^5(i) - z_l^5(i)\}\end{array}\right]}{Q_{55}(i)}} \quad (32)$$

Similarly, $k_{sy}^2$ is defined with $Q_{55}$ replaced by $Q_{44}$.

The FSDT models with the more general values of $k_{sx}^2$ and $k_{sy}^2$ are called model 11 and model 12 for the symmetric and unsymmetric laminates, respectively.

## 5. CONCLUSIONS

A unified general formulation of all higher-order theories has been presented for geometrically nonlinear response of cross-ply composite and sandwich plates based on a single polynomial expansion of displacement in the thickness coordinate. It includes the existing ten models as special cases. The governing displacement equations for linear static, free-vibration response, and for buckling under inplane static load, have been derived. The stiffness, inertia, geometric stiffness matrices, and the load vector for simply supported rectangular plate have been developed using Navier's solution. A general purpose program has been developed for all higher-order theories of a laminated plate.

## REFERENCES


1. Lo, K.H.; Christensen, R.M. & Wu, E.M. A higher-order theory of plate deformation, Part 1: Homogeneous plates. *ASME J. Appl. Mech.*, 1977, **44**(4), 663-68.

2. Lo, K.H.; Christensen, R.M. & Wu, E.M. A higher-order theory of plate deformation, Part II: Homogeneous plates. *ASME J. Appl. Mech.*, 1977, **44**(4), 669-76.

3. Kant, T. Numerical analysis of thick plates. *Comput. Meth. Appl. Mech. Eng.*,1982, **31**, 1-18.

4. Reddy, J.N. A simple higher-order theory for laminated composite plates. *ASME J. Appl. Mech.*, 1984, **51**, 745-52.

5. Reddy, J.N. & Phan, N.D. Stability and vibration of isotropic and laminated plates according to higher-order shear deformation theory. *J. Sound Vibration*, 1985, **98**, 157-70.

6. Senthilnathan, N.R.; Lim, K.H.; Lee, K.H. & Chow, S.T. Buckling of shear deformable plates. *AIAA Journal*, 1987, **25**(9), 1268-271.

7. Pandya, B.N. & Kant, T. Higher-order shear deformation theories for flexure of sandwich plates–finite-element evaluations. *Int. J. solids Struct.*, 1988, **24**(12)}, 1267-286.

8. Kant, T. & Manjunatha, B.S. An unsymmetric FRC laminate $C^o$ finite-element model with 12 degrees of freedom per node. *Engineering with Computers*, 1988, **5**(3), 300-08.

9. Noor, A.K. & Burton, W.S. Assessment of shear deformation theories for multilayered composite plates. *Appl. Mech. Rev.*, 1989, **42**(1), 1-13.

10. Srinivas, S.; Joga, C.V. & Rao, A.K. An exact analysis for vibration of simply supported homogeneous and laminated thick rectangular plates. *J. Sound Vibration*, 1970, **12**, 187-99.

11. Srinivas, S. & Rao, A.K. Bending vibration and buckling of simply supported thick orthotropic plates and laminates. *Int. J. Solids Struct.*, 1970, **6**, 1463-481.

12. Noor, A.K. Free vibrations of multilayered composite plates. *AIAA Journal*, 1973, **11**, 1038-39.

13. Swaminathan, K. Analytical evaluation of higher-order theories for stress, free vibration, and buckling analysis of cross-ply composite and sandwich plates. PhD Thesis, Dept of Civil Engg, IIT Bombay, Mumbai, 1999.

14. Kant, T. & Swaminathan, K. Analytical solutions for free vibration of laminated composite and

sandwich plates based on a higher-order refined theory. *Composite Structures*, 2001, **53**, 73-85.

15. Pagano, N.J. Exact solutions for rectangular bidirectional composites and sandwich plates. *J. Compo. Mater.*, 1970, **4**(1), 20-34.

16. Pagano, N.J. & Hatfield, S.J. Elastic behaviour of multilayered bidirectional composites. *AIAA Journal*, 1972, **10**(7), 931-33.

17. Noor, A.K. Stability of multilayered composite plates. *Fibre Sci. Tech.*, 1975, **8**(2), 81-89.

**Contributors**

**Dr GP Dube** is working as Professor in the Applied Mechanics Dept of Indian Institute of Technology (IIT) Delhi, New Delhi. His areas of interest are mechanics of solids and smart structures.

**Dr S Kapuria** is working as Associate Professor in the Applied Mechanics Dept of the IIT Delhi. His areas of interest are mechanics of solids, smart structures, and meshless computation.

**Dr PC Dumir** is working as Professor in the Applied Mechanics Dept of the IIT Delhi. His areas of interest are mechanics of solids and smarts structures.