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A Closed-form Solution to Finite Bending of a Compressible Elastic-perfectly Plastic Rectangular Block*

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ABSTRACT

The self-consistent Eulerian rate-type elastoplastic model based on the logarithmic rate is used to study finite bending of a compressible elastic-perfectly plastic rectangular block. It is found that an explicit closed-form solution for this typical inhomogeneous finite deformation mode may be available in a general case of compressible deformation with a stretch normal to the bending plane, where the maximum circumferential stretch at the outer surface serves as an independent parameter. Expressions are given for the bending angle, the bending moment, the outer and the inner radii, and the radii of the two moving elastic-plastic interfaces, etc. The exact stress distribution on any circumferential cross-section of the deformed block is accordingly determined.

Keywords: Finite bending, closed-form solution, elasto-plastic model, finite deformation, compressible deformation, elastic-plastic interface

1. INTRODUCTION

Finite bending of a rectangular block, such as a metal sheet, is a typical inhomogeneous finite deformation mode of practical interest. Actually, it does not appear to be easy to work out a complete closed-form solution for the stress distribution of this problem. Except for some particular cases, e.g., the incompressibility case, exact results seem rare even for inhomogeneous finite elastic deformations. For plastic bending at finite strain, results are mainly concerned with the cases of ideal plasticity without elastic deformation effects. In this aspect, the first complete analytical solution was presented by Hill[1] and Lubahn and Sachs[2]. Proksa[3] extended the study by incorporating work-hardening effect. Plane-strain bending of rigid plastic notched bars were investigated by a number of researchers[4-8] and the study has been extended to include work-hardening effect[9].

If elastic deformation effect is considered, the analysis would be difficult. It seems that the first satisfactory analytical results for elastoplastic bending were supplied by Boer[10,11], *et al.* for the case of incompressible deformations. Numerical results were provided by Bruhns[12], *et al.* for compressible elastic-perfectly plastic materials and by Bruhns[13,14], *et al.* for compressible elastoplastic materials with work-hardening behaviour. In these analyses, one of the essential points was to use Hencky's logarithmic strain measure[15-18], and a simple isotropic finite hyperelastic equation suggested by Hencky[15].

Recently, a new Eulerian rate-type model of finite elastoplasticity has been developed by Bruhns[19],

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* Dedicated to Professor Narinder K Gupta on his 60th birthday

et al. by virtue of Hencky's strain measure technique and the newly discovered[20] logarithmic rate $\boldsymbol{\tau}^{\circ\log}$. Unlike many other known models proposed earlier, the rate equation of hypoelastic-type for characterising elastic behaviour incorporated is self-consistent, i.e., it is exactly integrable to deliver an isotropic finite hyperelastic equation.

The new, self-consistent Eulerian rate-type model of finite elastoplasticity is used to study finite bending of a compressible elastic-perfectly plastic rectangular block with a stretch normal to the bending plane. In terms of the maximum circumferential stretch at the outer surface, closed-form expressions for the bending angle, the bending moment, the outer and the inner radii, and the radii of the two moving elastic-plastic interfaces are presented. The exact stress distribution on any circumferential section of the deformed block is determined.

2. ELASTIC-PERFECTLY PLASTIC MODEL BASED ON LOGARITHMIC RATE

Commonly used Eulerian rate-type formulations* of finite-strain elastoplasticity are based on the additive decomposition of the Eulerian stretching $\boldsymbol{D}$

$$\boldsymbol{D} = \boldsymbol{D}^e + \boldsymbol{D}^{ep} \tag{1}$$

The elastic stretching $\boldsymbol{D}^e$ is characterised by a rate constitutive equation

$$\boldsymbol{D}^e = \mathbf{C} : \boldsymbol{\tau}^{\circ} \text{ with } \quad \boldsymbol{\tau} = J\boldsymbol{\sigma}, \quad J = \det \boldsymbol{F} \tag{2}$$

where, $\boldsymbol{\tau}^{\circ}$ is an objective rate of the Kirchhoff stress $\boldsymbol{\tau}$, $\boldsymbol{\sigma}$ is the Cauchy stress, J is the specific volume ration, and $\boldsymbol{F}$ is the deformation gradient. The tangential elastic compliance tensor $\mathbf{C}$ is, in general, dependents on the current stress τ. It is widely assumed to be the classical isotropic elastic compliance tensor.

$$\mathbf{C} = \frac{1}{2G}\bar{\boldsymbol{I}} - \frac{1}{2G}\frac{\nu}{1+\nu}\boldsymbol{I} \otimes \boldsymbol{I} \tag{3}$$

Hence, Eqn (2) becomes

$$\boldsymbol{D}^e = \frac{1}{2G}\overset{\circ}{\boldsymbol{\tau}} - \frac{1}{2G}\frac{\nu}{1+\nu}(\mathrm{tr}\dot{\boldsymbol{\tau}})\boldsymbol{I} \tag{4}$$

where $\boldsymbol{I}$ and $\bar{\boldsymbol{I}}$ are the 2nd and 4th-order identity tensors, respectivley. G and ν are the shear modulus and the Poisson ratio, respectively, evaluated at infinitesimal deformations.

The rate of $\boldsymbol{\tau}$ in Eqn (4) may be chosen from among a variety of objective rates, such as Oldroyd rate, Cotter-Rivlin rate, Truesdell rate, Zaremba-Jaumann rate, and Green-Naghdi rate. It was demonstrated by Simond Pister[25] that none of the commonly used objective rates makes Eqn (4) exactly integrable to deliver an elastic relation, i.e., these are incompatible with the notion of elasticity.

Recently, these authors introduced the definition of the logarithmic tensor rate[20]. It has been shown[19, 26-28] that the elastic rate [Eqn (4)] with the logarithmic stress rate $\boldsymbol{\tau}^{\circ} \equiv \boldsymbol{\tau}^{\circ\log}$ is self-consistent, i.e., it is exactly integrable to deliver an isotropic finite hyperelastic equation; furthermore, only Eqn (4) with $\boldsymbol{\tau}^{\circ} \equiv \boldsymbol{\tau}^{\circ\log}$ can fulfil the foregoing self-consistency requirement.

$\boldsymbol{D}^{ep}$, the inelastic part of $\boldsymbol{D}$, is governed by a flow rule. For elastic-perfectly plastic materials with a yield function $f = \bar{f}(\boldsymbol{\tau})$, the associated flow rule yields:

$$\boldsymbol{D}^{ep} = \dot{\rho}\frac{\partial f}{\partial \boldsymbol{\tau}} \tag{5}$$

The plastic multiplier $\dot{\rho}$ is determined by the consistency condition $\dot{f} = 0$ for plastic flow. It is non-vanishing only for the loading case. Further details about ρ and the loading criterion will not be supplied, for there will be no need for these details in the subsequent analysis.

For this analysis it is convenient to use Tresca yield condition[12-14]

$$f = \bar{f}(\boldsymbol{\tau}) = \frac{\tau_1 - \tau_2}{2} - k = 0 \tag{6}$$

where the constant k is the yielding shear stress and τ_1 and τ_2 are the greatest and the smallest

* It seems that such a journal formulation was definitely proposed by Hill[21] and Lehmann[22] related references may be found in references 23 and 24.

of the three principal values of the Kirchhoff stress tensor τ, respectivley.

Equations (1), (2) and (4) for $\tau^{\circ} \equiv \tau^{\circ \log}$ and Eqn (5) together yield

$$\tau^{\circ \log} = \frac{2G\nu}{1-2\nu}(\mathrm{tr}\boldsymbol{D})\boldsymbol{I} + 2G\boldsymbol{D} - 2G\dot{\rho}\frac{\partial f}{\partial \tau} \quad (7)$$

3. KINEMATICS & THE EQUATIONS OF EQUILIBRIUM

A general description of finite bending of a rectangular block can be found[29,30]. It does not seem to be easy to work out a complete explicit solution even for a simple form of compressible elastic material(see, e.g., the treatment given in Ref [30] for harmonic materials). Closed-form solutions, indeed, are rare for inhomogeneous finite compressible deformations of realistic materials. In the context of classical isotropic infinitesimal elasticity, a complete analysis for pure bending of a solid cone has been achieved only very recently[31]. The incompressibility condition may result in a substantial simplification[32]. For finite bending of a block, the forms of the radial and circumferential displacement components may be statically determined in the case of incompressibility, whereas in the case of compressible deformation, the radial displacement component assumes a general unknown form which is, in a complicated manner, coupled with the material property, as will be seen below.

A rectangular block with undeformed lengths $2l_0$, $2t_0$ and $2h_0$ in its three mutually perpendicular directions, respectivley has been considered. A fixed rectangular Cartesian coordinate system $OXYZ$ is introduced (Fig. 1). It has been assumed that the block is deformed into a sector of a circular cylindrical

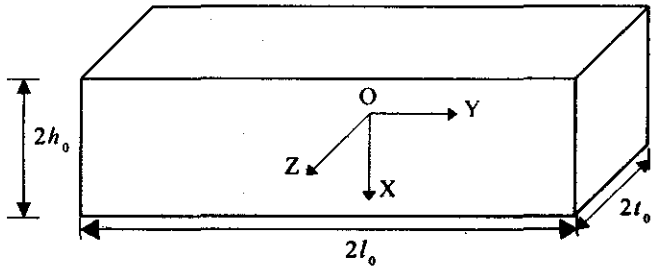

Figure 1. An undeformed rectangular block

tube (Fig. 2). A cylindrical coordinate system $or\theta z$ was choosen. Then a plane X = const. in the block in Fig. 1 becomes a sector of a circular cylindrical surface r = const. in Fig. 2, a plane Y = const. becomes a plane θ = const. and a plane Z = const. becomes a plane z = const.

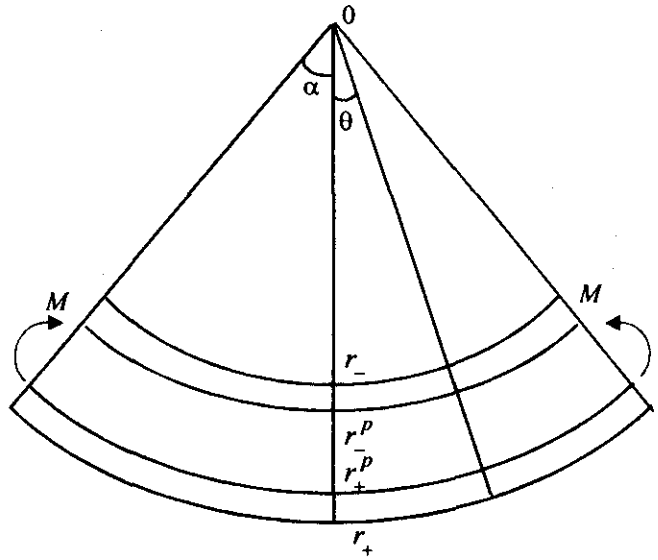

Figure 2. Deformed state of a rectangular block by finite bending.

The bending of the block (Fig. 1) into the sector (Fig. 2) may be described by

$$r = \bar{r}(X), \qquad \theta = \frac{\alpha}{l_0}Y, \qquad z = \lambda Z \quad (8)$$

where 2α is the bending angle and λ is the stretch normal to the bending plane[30]. λ is regarded as a given quantity.

Let $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$ be a reference rectangular Cartesian basis in the directions of OX-, OY- and OZ-axes, and let $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$ be a current cylindrical polar basis where $\boldsymbol{e}_z = \boldsymbol{E}_3$ and $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are in the radial and circumferential directions, respectivley in Fig. 2. Then the deformation gradient $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = \lambda_r \boldsymbol{e}_r \otimes \boldsymbol{E}_1 + \lambda_\theta \boldsymbol{e}_\theta \otimes \boldsymbol{E}_2 + \lambda \boldsymbol{e}_z \otimes \boldsymbol{E}_3 \quad (9)$$

where $\lambda_r = r'$ denotes the radial stretch, and $\lambda_\theta = \alpha r / l_0$ denotes the circumferential stretch. Here and henceforth, a prime(′) means the differentiation wrt X.

The left Cauchy-Green tensor $\boldsymbol{B}$ is of the form

$$\boldsymbol{B} = \lambda_r^2 \boldsymbol{e}_r \otimes \boldsymbol{e}_r + \lambda_\theta^2 \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \lambda^2 \boldsymbol{e}_z \otimes \boldsymbol{e}_z \quad (10)$$

From Eqn (10) it may become clear that the radial and circumferential stretches λ_r and λ_θ are the two principal stretches and that their corresponding principal axes are in the radial and circumferential directions $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$.

It is straightforward to obtain the Hencky's strain tensor $\boldsymbol{h}$, the volume ratio J, the velocity gradient $\boldsymbol{L}$, the stretching $\boldsymbol{D}$ and the vorticity tensor $\boldsymbol{W}$ as

$$\begin{aligned} \boldsymbol{h} &= \frac{1}{2} \ln \boldsymbol{B} \\ &= \ln \lambda_r \boldsymbol{e}_r \otimes \boldsymbol{e}_r + \ln \lambda_\theta \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \ln \lambda \boldsymbol{e}_z \otimes \boldsymbol{e}_z \end{aligned} \quad (11)$$

$$J = \det \boldsymbol{F} = \lambda \lambda_r \lambda_\theta \quad (12)$$

$$\begin{aligned} \boldsymbol{L} &= \dot{\boldsymbol{F}} \boldsymbol{F}^{-1} \\ &= \frac{\dot{\lambda}_r - \dot{\theta} \lambda_\theta}{\lambda_r} \boldsymbol{e}_r \otimes \boldsymbol{e}_r + \frac{\dot{\lambda}_\theta - \dot{\theta} \lambda_r}{\lambda_\theta} \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta \end{aligned} \quad (13)$$

$$\boldsymbol{D} = \frac{1}{2}\left(\boldsymbol{L} + \boldsymbol{L}^T\right) = \boldsymbol{L} \quad (14a)$$

$$\boldsymbol{W} = \frac{1}{2}\left(\boldsymbol{L} - \boldsymbol{L}^T\right) = 0 \quad (14b)$$

The Cauchy stress σ is always coaxial with B

$$\boldsymbol{\sigma} = \sigma_r \boldsymbol{e}_r \otimes \boldsymbol{e}_r + \sigma_\theta \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \sigma_z \boldsymbol{e}_z \otimes \boldsymbol{e}_z \quad (15)$$

where σ_r, σ_θ and σ_z are the principal stress components in the radial, circumferential, and in oz-axis directions, respectively.

In the absence of body forces, the equation of equilibrium is given by

$$\operatorname{div} \boldsymbol{\sigma} = 0$$

Formulating the latter with reference to the coordinate system $or\theta z$, one can derive three equations. Of these, only one is non-trivial and of the form

$$\sigma'_r + \frac{\lambda'_\theta}{\lambda_\theta}\left(\sigma_r - \sigma_\theta\right) = 0 \quad (16)$$

It has been assumed that the bending moment M is gradually increasing, i.e., there is no unloading. Let r_+ and r_- be the outer and the inner radii of the deformed block, respectively as shown in Fig. 2. The process of deformation is as follows:

- When the bending moment M does not exceed a threshold value M_0, the whole region of the block will be elastically deformed.
- When $M = M_0$, the initial yielding starts at the outer and the inner surfaces, $r = r_\pm$.
- When $M > M_0$, there are two plastic regions $r_+^p \le r \le r_+$ and $r_- \le r \le r_-^p$ with an elastic region $r_-^p \le r \le r_+^p$ in between.

In the succeeding sections, the elastic and elastic-plastic solutions, respectively will be determined. To find out a compressible finite deformation solution, one has to cope with strongly nonlinear coupling problems with two moving elastic-plastic interfaces. Among the unknowns included are the bending moment M, the bending angle 2α, the outer and inner radii r_+ and r_-, the maximum and the minimum circumferential stretches $\lambda_\theta^\pm$ at the outer and the inner surfaces $r = r_\pm$, as well as the the current radii $r_\pm^p$ of the two moving elastic-plastic interfaces and the circumferential stretches $\lambda_\theta^{p\pm}$ at $r = r_\pm^p$. A basic fact is that one of these unknowns determines all the others. Hence, one may choose one of the foregoing unknowns as an independent parameter and regard any of the others as a function of this chosen independent parameter. This study shows that it is possible to work out an explicit closed-form solution by selecting either of the maximum or minimum circumferential stretches $\lambda_\theta^\pm$ at the outer and the inner surfaces $r = r_\pm$ as an independent parameter. It is pointed out that the choice of independent parameter is crucial to achieving the goal.

4. ELASTIC SOLUTION

When the bending moment M does not exceed a threshold value M_0, the block only experiences elastic deformation. Because there is no plastic deformation, i.e., $\boldsymbol{D}^{ep} = \boldsymbol{0}$, $\dot{\rho} = 0$, $\boldsymbol{D}^e = \boldsymbol{D}$, Eqn (7) reduces to

$$\boldsymbol{\tau}^{\circ\log} = \frac{2G\nu}{1-2\nu}(\mathrm{tr}\boldsymbol{D})\boldsymbol{I} + 2G\boldsymbol{D} \tag{17}$$

Utilising the kinematical relation $\boldsymbol{D} = \boldsymbol{h}^{\circ\log}$, the path-independent integration of Eqn (17) is derived as follows:

$$\boldsymbol{\tau} = \frac{2G\nu}{1-2\nu}(\mathrm{tr}\boldsymbol{h})\boldsymbol{I} + 2G\boldsymbol{h} \tag{18}$$

This is exactly the isotropic finite elastic equation introduced by Hencky[15] more than 70 years ago. It is hyperelastic or elastic in Green's sense[33]. Hencky's elasticity model[18] has been widely used in finite inelastic modelling and related finite-element simulations[10-14,34-43]. Anand[44-45] found that for a number of typical deformation modes, Hencky's model[18] is able to fit experimental data better than several other known models. Also, rigorous theoretical foundation of Hencky's model[18] has been examined by Bruhns[46], *et al.* concerning certain well-established constitutive inequalities, including Baker-Ericksen inequalities, Hill's inequalities wrt Hencky strain, as well as Legendre-Hadamard inequalities or ellipticity, etc.

The finite elastic bending for $M < M_0$ is just that of a rectangular block made of Hencky's elastic material defined by Eqn (18). A closed-form solution has been derived[47]. Some relevant results are given below.

Substituting Eqns (11) and (15) into Eqn (18), one gets:

$$\tau_r = J\sigma_r = \frac{2G\nu}{1-2\nu}\left((1-\nu)\ln\lambda_r + \nu(\ln\lambda_\theta + \ln\lambda)\right)$$

$$\tau_\theta = J\sigma_\theta = \frac{2G\nu}{1-2\nu}\left((1-\nu)\ln\lambda_\theta + \nu(\ln\lambda_r + \ln\lambda)\right)$$

$$\tau_z = J\sigma_z = \frac{2G\nu}{1-2\nu}\left((1-\nu)\ln\lambda + \nu(\ln\lambda_0 + \ln\lambda_r)\right) \tag{19}$$

Then, with Eqns (12), (16) and (19) and the equality

$$J'/J = \lambda_r'/\lambda_r + \lambda_\theta'/\lambda_\theta \tag{20}$$

the governing equation for elastic deformation has been derived as follows:

$$\frac{\lambda_r'}{\lambda_r} + \frac{\lambda_\theta'}{\lambda_\theta}\frac{(1-\nu)\ln\lambda_\theta + \nu\ln\lambda_r + \nu\ln\lambda - \nu}{\nu\ln\lambda_\theta + (1-\nu)\ln\lambda_r + \nu\ln\lambda + \nu - 1} = 0 \tag{21}$$

The boundary conditions are given by $\sigma_r\big|_{r=r_\pm} = 0$. Herewith and with Eqn (21) the following explicit results have been derived[47] for the bending angle 2α, the bending moment M per unit length, the principal stresses σ_r, σ_θ, σ_z, and the coordinate X corresponding to bending angle 2α and radius r.

$$2\alpha\frac{h_0}{l_0} = \frac{1}{e}\int_{\lambda_0^-}^{\lambda_0^+}(\lambda\lambda_0)^{\frac{\nu}{1-\nu}}e^{\sqrt{\Phi}}d\lambda_0 \tag{22}$$

$$\frac{M}{Gh_0^2} = 8e\lambda^{\frac{-1}{1-\nu}}\frac{\displaystyle\int_{\lambda_\theta}^{\lambda_0^+}\lambda_0^{\frac{\nu}{1-\nu}}e^{\sqrt{\Phi}}\left(\frac{\ln\lambda_0 + \nu\ln\lambda}{1-\nu} + \nu\frac{1-\sqrt{\Phi}}{1-2\nu}\right)d\lambda_0}{\left(\displaystyle\int_{\lambda_0}^{\lambda_0^+}\lambda_0^{\frac{\nu}{1-\nu}}e^{\sqrt{\Phi}}d\lambda_0\right)^2} \tag{23}$$

$$\sigma_r = 2G(\lambda\lambda_0)^{\frac{1-2\nu}{1-\nu}}e^{\sqrt{\Phi}-1}(1-\nu)\frac{1-\sqrt{\Phi}}{1-2\nu} \tag{24}$$

$$\sigma_\theta = 2G(\lambda\lambda_\theta)^{\frac{1-2\nu}{1-\nu}}e^{\sqrt{\Phi}-1}\left(\frac{\ln(\lambda^\nu\lambda_0)}{1-\nu} + \nu\frac{1-\sqrt{\Phi}}{1-2\nu}\right) \tag{25}$$

$$\sigma_z = 2G(\lambda\lambda_\theta)^{-\frac{1-2\nu}{1-\nu}} e^{\sqrt{\Phi}-1}\left(\frac{\ln(\lambda\lambda_\theta^\nu)}{1-\nu} + \nu\frac{1-\sqrt{\Phi}}{1-2\nu}\right) \tag{26}$$

$$\frac{X}{h_0} = -1 + 2\frac{\int_{\lambda_\theta^-}^{\lambda_\theta^+} \lambda_\theta^{\frac{\nu}{1-\nu}} e^{\sqrt{\Phi}} d\lambda_\theta}{\int_{\lambda_\theta^-}^{\lambda_\theta^+} \lambda_\theta^{\frac{\nu}{1-\nu}} e^{\sqrt{\Phi}} d\lambda_\theta} \tag{27}$$

The maximum and minimum circumferential stretches $\lambda_\theta^\pm$ are related by

$$\lambda_\theta^+\lambda_\theta^- = \lambda^{-2\nu} \tag{28}$$

and Φ is used to denote the function

$$\Phi = 1 + \frac{1-2\nu}{(1-\nu)^2}\left(\gamma^2 - \ln^2(\lambda_\theta\lambda^\nu)\right) \tag{29a}$$

$$\gamma^2 = \ln^2(\lambda_\theta^+\lambda^\nu) = \ln^2(\lambda_\theta^-\lambda^\nu) \tag{29b}$$

As will be seen, relation Eqn (28) is universal for the whole process of elastic and elastic-plastic bending.

5. ELASTIC-PLASTIC SOLUTION

The elastic region and the two plastic regions have been investigated, separately. First, the Tresca yield condition [Eqn (6)] need to be formulated for finite bending in a specific form. As indicated by Bruhns[12], *et al.* and Bruhns[13], in the plane strain case with stretch $\lambda = 1$, the magnitude of the circumferential stress σ_θ is far greater than that of the radial stress σ_r and $(\sigma_r+\sigma_\theta)/2$ is close to the stress σ_z normal to the bending plane. It is expected that these facts remain trùe, whenever the stretch 3λ is sufficiently close to 1. As a result, Eqn (6) assumes the form

$$f = \bar{f}(\tau) = \frac{\tau_\theta - \tau_r}{2} - k_\pm = 0 \tag{30}$$

where $k_+ = k$, $k_- = -k$

Here and hereafter, the indices + and – are associated with the outer plastic region $r_+^p \le r \le r_+$ and the inner plastic region $r_- \le r \le r_-^p$, respectivley.

5.1 Initial Yielding, Elastic Region & Interface Conditions

First, the case has been analysed when the initial yielding starts at the outer and inner surfaces ($M = M_0$). From the yield condition [Eqn (30)], $\sigma_r = 0|_{r=r_\pm}$ yield and Eqn (19) one derives for $r = r_\pm$

$$\tau_\theta - \tau_r = 2k_\pm$$

$$(1-\nu)\ln\lambda_r + \nu(\ln\lambda_\theta + \ln\lambda) = 0 \tag{31}$$

$$\lambda_{\theta 0}^+ = \lambda^{-\nu} e^{(1-\nu)\frac{k}{G}} \tag{32a}$$

$$\lambda_{\theta 0}^- = \lambda^{-\nu} e^{-(1-\nu)\frac{k}{G}} \tag{32b}$$

The corresponding bending angle $2\alpha_0$, the bending moment M_0 and the stresses may accordingly be obtained using Eqns (22)-(26) with Eqn (32). $\lambda_{\theta 0}^+$ and $\lambda_{\theta 0}^-$ obey relation [Eqn (28)], i.e., the outer and the inner surfaces indeed enter into the initial yielding state simultaneously.

For $M > M_0$, the elastic region, specified by $r_-^p \le r \le r_+^p$, lies between the two plastic regions. The governing equation for the elastic region remains Eqn (21). Across the two elastic-plastic interfaces at $r = r_\pm^p$, the radial stress σ_r and the two stretches λ_r and λ_θ should be continuous and the yield condition [Eqn (30)] should be met. Hence, one has

$$\sigma_\theta^e - \sigma_r^e = k_\pm \tag{33a}$$

$$\lambda_r^e = \lambda_r^p \tag{33b}$$

$$\lambda_\theta^e = \lambda_\theta^p \tag{33c}$$

$$\sigma_r^e = \sigma_r^p \tag{33d}$$

The superscripts e and p are associated with the elastic and plastic regions, respectively. Conditions[Eqns {33(a)-(d)}] imply that the three principal stresses are also continuous across the two elastic-plastic interfaces.

5.2 Plastic Regions

Within the two plastic regions the derivative of the yield function is given by

$$\frac{\partial f}{\partial \boldsymbol{\tau}} = \left(\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta - \boldsymbol{e}_r \otimes \boldsymbol{e}_r\right)/2 \tag{34}$$

Equations (13) and [14(a)] show that the Cauchy-Green tensor $\boldsymbol{B}$ and the stretching $\boldsymbol{D}$ are coaxial. This and Eqn [14(b)] result in the simplification

$$\overset{\circ}{\boldsymbol{\tau}}{}^{\log} = \dot{\boldsymbol{\tau}} = \dot{\tau}_r \boldsymbol{e}_r \otimes \boldsymbol{e}_r + \dot{\tau}_\theta \boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + \dot{\tau}_z \boldsymbol{e}_z \otimes \boldsymbol{e}_z + \theta\left(\dot{\tau}_r - \tau_\theta\right)\left(\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_r\right) \tag{35}$$

Then, the elastoplastic [Eqn (7)] reduces to

$$\dot{\boldsymbol{\tau}} = \frac{2Gv}{1-2v}(\mathrm{tr}\boldsymbol{D})\boldsymbol{I} + 2G\boldsymbol{D} - 2G\rho\frac{\partial f}{\partial \boldsymbol{\tau}} \tag{36}$$

Hence, Eqns (12), (14) and Eqns (34)-(36) generate

$$\dot{\tau}_r = \frac{2Gv}{1-2v}\overline{\ln J} + 2G\boldsymbol{D}\frac{\lambda_r - \dot{\theta}\lambda_\theta}{\lambda_r} + G\rho \tag{37a}$$

$$\dot{\tau}_z = \frac{2Gv}{1-2v}\overline{\ln J} \tag{37b}$$

$$\dot{\tau}_\theta = \frac{2Gv}{1-2v}\overline{\ln J} + 2G\boldsymbol{D}\frac{\dot{\lambda}_\theta - \dot{\theta}\lambda_r}{\lambda_\theta} + G\rho \tag{37c}$$

$$\dot{\theta}\left(\tau_r - \tau_\theta\right) = 0 \tag{37d}$$

Equilibrium Eqns (16), (30) and Eqn 37 (a) to (d) constitute the governing equations for the deformations and stresses within the two plastic regions. In addition to the interface conditions [Eqn 33(a)-(d)] the boundary conditions $\left.\sigma_r\right|_{r=r_\pm} = 0$ should be met.

5.3 The Explicit Closed-form Solution

An elastic-plastic solution may be derived by means of four procedures.

(i) The plastic regions specified by $r_- \le r \le r_-^p$ and $r_+^p \le r \le r_+$ is analysed. Equations (30) and 37 (d) yield $\dot{\theta} = 0$. From this and Eqn 37 (a)-(c) one deduces

$$\dot{\tau}_r + \dot{\tau}_\theta = \frac{2G}{1-2v}\overline{\ln J}$$

$$\dot{\tau}_r + \dot{\tau}_\theta + \dot{\tau}_z = 2G\frac{1+v}{1-2v}\overline{\ln J}$$

Thus, one arrives at

$$\tau_z = \frac{2G}{1-2v}\ln\left(\lambda^{1-v}\lambda_r^\theta\lambda_0^v\right) \tag{38a}$$

$$\tau_r + \tau_\theta = \frac{2G}{1-2v}\ln\left(\lambda_r\lambda_0\lambda^{2v}\right) \tag{38b}$$

Eqns (30) and [38(b)] together produce

$$\tau_r = J\sigma_r = \frac{G\ln\left(\lambda_r^\theta\lambda_0^v\lambda^{2v}\right)}{1-2v} - k_\pm \tag{39a}$$

$$\tau_\theta = J\sigma_\theta = \frac{G\ln\left(\lambda_r\lambda_0\lambda^{2v}\right)}{1-2v} + k_\pm \tag{39b}$$

Substituting Eqn (39) into Eqn (16) and using Eqn (20), one arrives at

$$\frac{\lambda_r'}{\lambda_r} + \frac{\lambda_\theta'}{\lambda_\theta}\frac{\ln\left(\lambda_r\lambda_\theta\lambda^{2v}\right) - 1 + (1-2v)k_\pm/G}{\ln\left(\lambda_r\lambda_\theta\lambda^{2v}\right) - 1 - (1-2v)k_\pm/G} = 0 \tag{40}$$

With $d\ \ln\lambda_r = d\lambda_r/\lambda_r$ and $d\ \ln\lambda_\theta = d\lambda_\theta/\lambda_\theta$ from the integration of Eqn (40), one derives

$$\ln^2\left(\lambda_r\lambda_\theta\right) + 2a\ln\lambda_r + 2\ln\left(\lambda_r^a\lambda_\theta^b\right) = c \tag{41}$$

where c is an integration constant and

$$a = 2v\ln\lambda - 1 - (1-2v)k_{\pm}/G$$

$$b = 2v\ln\lambda - 1 + (1-2v)k_{\pm}/G$$

Equation (41) generates

$$\ln\lambda_r = -\ln\lambda_\theta - a + \delta\sqrt{2(a-b)\ln\lambda_\theta + a^2 + c}$$

where δ may be either 1 or -1. Using the boundary condition $\sigma_r|_{r=r_\pm}$, Eqn [39(a)] and Eqn (41) with $r = r_\pm$, one can determine δ and c and get

$$\ln\lambda_r = 1 + (1-2v)\frac{k_\pm}{G} - \ln\left(\lambda^{2v}\lambda_\theta\right) - \sqrt{\psi_\pm} \tag{42}$$

where

$$\psi_\pm = 1 - 4(1-2v)\frac{k_\pm}{G}\ln\frac{\lambda_\theta}{\lambda_\theta^\pm} \tag{43}$$

where $\lambda_\theta^\pm$ are the maximum and the minimum circumferential stretches at $r = r_\pm$, respectivley. Thus, the principal stresses at the two plastic regions are given by

$$\sigma_r = G\xi_\pm e^{\sqrt{\Psi_\pm}}\frac{1-\sqrt{\Psi_\pm}}{1-2v} \tag{44}$$

$$\sigma_\theta = G\xi_\pm e^{\sqrt{\Psi_\pm}}\left(\frac{1-\sqrt{\Psi_\pm}}{1-2v} + 2\frac{k_\pm}{G}\right) \tag{45}$$

$$\sigma_z = 2G\xi_\pm e^{\sqrt{\Psi_\pm}}\left(v\frac{k_\pm}{G} + (1+v)\ln\lambda + v\frac{1-\sqrt{\Psi_\pm}}{1-2v}\right) \tag{46}$$

where

$$\xi_\pm = \lambda^{-(1-2v)}e^{-(1-2v)\frac{k_\pm}{G} - 1} \tag{47}$$

Besides, utilising Eqn (42) and the relation

$$\lambda_r = \frac{l_0}{\alpha}\frac{d\lambda_\theta}{dX} \tag{48}$$

one infers for the two plastic regions

$$(X \mp h_0)\frac{\alpha}{l_0} = \lambda\xi_\pm \int_{\lambda_\theta^\pm}^{\lambda_\theta} \lambda_\theta e^{\sqrt{\Psi_\pm}} d\lambda_\theta \tag{49}$$

(ii) The integration of Eqn (21) has been worked out for the elastic region $r_-^p \le r \le r_+^p$. Using $d\ln\lambda_r = d\lambda_r/\lambda_r$ and $d\ln\lambda_\theta = d\lambda_\theta/\lambda_\theta$, one may integrate Eqn (21)

$$v\ln\lambda_\theta . \ln\lambda_r + \frac{1-v}{2}\left(\ln^2\lambda_\theta + \ln^2\lambda_r\right) + (v\ln\lambda - v)\ln\lambda_\theta + (v\ln\lambda + v - 1)\ln\lambda_r = C \tag{50}$$

Hence, one obtains

$$\ln\lambda_r = 1 - \frac{\ln(\lambda\lambda_\theta)^v}{1-v} + \delta\sqrt{\overline{C} - \frac{1-2v}{(1-v)^2}\ln^2\left(\lambda^v\lambda_\theta\right)} \tag{51}$$

Herein, the sign $\delta = 1$ or $\delta = -1$ and the integration constant $\overline{C}$ are to be determined.

(iii) The interface conditions [Eqn 33(a)-(d)] have been considered

Using the yield condition [Eqn (33a)] and Eqn (19) and (51), one may determine the constant $\overline{C}$ and the sign δ. The results are as follows:

$$\overline{C} = \eta_\pm^2 + \frac{1-2v}{(1-v)^2}\ln^2\left(\lambda^v\lambda_\theta^{p\pm}\right) \tag{52a}$$

$$\ln\lambda_r = 1 - \eta_\pm\sqrt{\Theta_\pm} - \frac{\ln(\lambda\lambda_\theta)^v}{1-v} \tag{52b}$$

where $\lambda_\theta^{p\pm} = \lambda_\theta|_{r=r_\pm^p}$. Here and henceforward

$$n_{\pm} = 1 + \frac{k_{\pm}}{G} - \frac{\ln\left(\lambda^{\nu}\lambda_{\theta}^{p\pm}\right)}{1-\nu} \tag{53}$$

$$\Theta_{\pm} = 1 - \frac{1-2\nu}{\eta_{\pm}^{2}(1-\nu)^{2}} \ln\frac{\lambda_{\theta}}{\lambda_{\theta}^{p\pm}} \ln\left(\lambda^{2\nu}\lambda_{\theta}^{p\pm}\lambda_{\theta}\right) \tag{54}$$

$\overline{C}$ given by Eqn [52(a)] should be the same for the two cases with ±, i.e.,

$$\eta_{+}^{2} + \frac{1-2\nu}{(1-\nu)^{2}} \ln^{2}\left(\lambda^{\nu}\lambda_{\theta}^{p+}\right) = \eta_{-}^{2} + \frac{1-2\nu}{(1-\nu)^{2}} \ln^{2}\left(\lambda^{\nu}\lambda_{\theta}^{p-}\right). \tag{55}$$

From the latter and Eqn (53), one infers that $\lambda_{\theta}^{p\pm}$ are related to each other by

$$\lambda_{\theta}^{p-} = \lambda^{-\nu} e^{\frac{1-\frac{k}{G}}{2} - \sqrt{\left(\ln\left(\lambda^{\nu}\lambda_{\theta}^{p+}\right) - \frac{1+\frac{k}{G}}{2}\right)^{2} + (1-2\nu)\frac{k}{G}}} \tag{56}$$

On the other hand, with the continuity conditions [Eqn (33)(b) and (c)] for the stretches λ_r and λ_θ and Eqns (42)-(43) and [Eqns (52(b)-54)} we have

$$1 - \frac{\ln\left(\lambda\lambda_{\theta}^{p\pm}\right)^{\nu}}{1-\nu} - \eta_{\pm}\sqrt{\Theta_{\pm}^{p}} = 1 - \ln\left(\lambda^{2\nu}\lambda_{\theta}^{p\pm}\right) + (1-2\nu)\frac{k_{\pm}}{G} - \sqrt{\Psi_{\pm}^{p}} \tag{57}$$

where

$$\Theta_{\pm}^{p} = \Theta_{\pm}\Big|_{r=r_{\pm}^{p}} = 1$$

$$\Psi_{\pm}^{p} = \Psi_{\pm}\Big|_{r=r_{\pm}^{p}} = 1 - 4(1-2\nu)\frac{k_{\pm}}{G}\ln\frac{\lambda_{\theta}^{p\pm}}{\lambda_{\theta}^{\pm}}$$

The last three equations produce

$$4(1-2\nu)\frac{k_{\pm}}{G}\ln\frac{\lambda_{\theta}^{p\pm}}{\lambda_{\theta}^{\pm}} = 1 - \left(\frac{1}{2}\ln\left(\lambda^{\nu}\lambda_{\theta}^{p\pm}\right) - \frac{1}{2}(1-\nu)\frac{k_{\pm}}{G} - 1\right)^{2} \tag{58}$$

From Eqn (58) with index +, one may derive an expression for λ_{θ}^{p+} in terms of the maximum circumferential stretch λ_{θ}^{+}. The result is as follows:

$$\lambda_{\theta}^{p+} = \lambda^{-\nu} e^{\frac{1}{2}\left(1+\frac{k}{G}\right) - \frac{1}{2}\sqrt{1+(1-2\nu)\frac{k}{G}\left(4\ln\left(\lambda^{\nu}\lambda_{\theta}^{+}\right) - (3-2\nu)\frac{k}{G} - 2\right)}} \tag{59}$$

Substitution of Eqn (59) into Eqn (56) yields

$$\lambda_{\theta}^{p-} = \lambda^{-\nu} e^{\frac{1}{2}\left(1-\frac{k}{G}\right) - \frac{1}{2}\sqrt{1+(1-2\nu)\frac{k}{G}\left(4\ln\left(\lambda^{\nu}\lambda_{\theta}^{+}\right) - (3-2\nu)\frac{k}{G} + 2\right)}} \tag{60}$$

Moreover, from Eqn (58) with index − , one can derive an expression for $\ln\lambda_{\theta}^{-}$ in terms of $\ln\lambda_{\theta}^{p-}$. Then, from this expression and Eqn (60) one derives

$$\lambda_{\theta}^{+}\lambda_{\theta}^{-} = \lambda^{-2\nu} \tag{61}$$

It may be verified that the continuity condition [Eqn 33(d)] for the radial stress σ_r can be satisfied, whenever the conditions [Eqn 33 (a)-(c)] are satisfied.

Equation (61) is just Eqn (28). This suggests that the maximum and minimum circumferential stretches λ_θ ± at the outer and the inner surfaces $r = r_{\pm}$ obey the same relation during the whole process of deformation, including both elastic and elastic-plastic deformations.

The principal stresses within the elastic region $r_{-}^{p} \le r \le r_{+}^{p}$ are given by

$$\sigma_r = 2G(\lambda\lambda_\theta)^{-\frac{1-2\nu}{1-\nu}} e^{-\left(1-\eta+\sqrt{\Theta_+}\right)} (1-\nu)\frac{1-\eta+\sqrt{\Theta_+}}{1-2\nu} \tag{62a}$$

$$\sigma_\theta = 2G(\lambda\lambda_\theta)^{-\frac{1-2\nu}{1-\nu}} e^{-\left(1-\eta+\sqrt{\Theta_+}\right)} \left(\frac{\ln\left(\lambda^{\nu}\lambda_\theta\right)}{1-\nu} + \nu\frac{1-\eta+\sqrt{\Theta_+}}{1-2\nu}\right) \tag{62b}$$

$$\sigma_z = 2G(\lambda\lambda_\theta)^{-\frac{1-2\nu}{1-\nu}} e^{-(1-\eta+\sqrt{\Theta_+})}\left(\frac{\ln(\lambda\lambda_\theta^\nu)}{1-\nu}+\nu\frac{1-\eta+\sqrt{\Theta_+}}{1-2\nu}\right) \quad (62c)$$

Moreover, using Eqns (48) and [52(b)] with the subscript +, one derives the reference coordinate X within the elastic region as follows:

$$\left(X - X_+^p\right)\frac{\alpha}{l_0} = \frac{1}{e}\int_{\lambda_\theta^{p+}}^{\lambda_\theta}(\lambda\lambda_\theta)^{\frac{\nu}{1-\nu}} e^{\eta_+\sqrt{\Theta_+}} d\lambda_\theta \quad (63)$$

Here and henceforth, $X_\pm^p$ has been used to stand for the reference coordinates corresponding to the current radii $r_\pm^p$ at the two elastic-plastic interfaces.

(iv) Finally, the bending angle 2α and the bending moment M are determined.

Setting $X = X_\pm^p$ and $\lambda_\theta = \lambda_\theta^{p\pm}$ in Eqn (49) and setting $X = X_-^p$ and $\lambda_\theta = \lambda_\theta^{p-}$ in Eqn (63), one obtains three integral expressions for $\left(X_\pm^p \mp h_0\right)\frac{\alpha}{l_0}$ and $\left(X_-^p - X_+^p\right)\frac{\alpha}{l_0}$. Utilising these three expressions, one arrives at

$$2\alpha\frac{h_0}{l_0} = \lambda\xi_+\int_{\lambda_\theta^{p+}}^{\lambda_\theta^+}\lambda_\theta e^{\sqrt{\Psi_+}}d\lambda_\theta + \frac{1}{e}\int_{\lambda_\theta^{p-}}^{\lambda_\theta^{p+}}(\lambda\lambda_\theta)^{\frac{\nu}{1-\nu}}e^{\eta_+\sqrt{\Theta_+}}d\lambda_\theta + \lambda\xi_-\int_{\lambda_\theta^-}^{\lambda_\theta^{p-}}\lambda_\theta e^{\sqrt{\Psi_-}}d\lambda_\theta \quad (64)$$

The bending moment M per unit length $M = \int_{-h_0}^{h_0} r\sigma_\theta\, dr = \left(\frac{l_0}{\alpha}\right)\int_{\lambda_\theta^-}^{\lambda_\theta^+}\lambda_\theta\sigma_\theta$ is obtained by Eqn (45) and Eqn [62(b)]

$$\frac{Mq^2}{Gh_0^2} = \xi_+\int_{\lambda_\theta^{p+}}^{\lambda_\theta^+}\lambda_\theta e^{\sqrt{\Psi_+}}\left(\frac{1-\sqrt{\Psi_+}}{1-2\nu}+2\frac{k}{G}\right)d\lambda_\theta + 2\lambda^{-\frac{1-2\nu}{1-\nu}}e^{-(1-\eta_+\sqrt{\Theta_+})}\int_{\lambda_\theta^{p-}}^{\lambda_\theta^{p+}}\lambda_\theta^{\frac{\nu}{1-\nu}}\left(\frac{\ln(\lambda^\nu\lambda_\theta)}{1-\nu}+\nu\frac{1-\eta_+\sqrt{\Theta_+}}{1-2\nu}\right)d\lambda_\theta + \xi_-\int_{\lambda_\theta^-}^{\lambda_\theta^{p-}}\lambda_\theta e^{\sqrt{\Psi_-}}\left(\frac{1-\sqrt{\Psi_-}}{1-2\nu}+2\frac{k}{G}\right)d\lambda_\theta \quad (65)$$

In the above, $q = 2\alpha\frac{h_0}{l_0}$ is given by Eqn (64); $\xi_\pm$ and $\Psi_\pm$ are given by Eqns (43) and (47); and η_+ and Θ_+ by Eqns (53)-(54) with index +.

5.4 Summary of Results

For any $\lambda_\theta^+ \in \left[1, \lambda_{\theta 0}^+\right]$, where $\lambda_{\theta 0}^+$ is evaluated by Eqn [32(a)], the whole block is elastically deformed. λ_θ^- is given by Eqn (61) or Eqn (28). The solution is determined by Eqns (22)-(29).

For any $\lambda_\theta^+ \in \left[\lambda_{\theta 0}^+, \infty\right]$ Eqn (61) determines λ_θ^-, too. $\lambda_\theta^{p\pm}$ are given by Eqns (59) and (60). Then, the bending angle 2α, the bending moment M and the three principal stresses σ_r, σ_θ and σ_z are determined by Eqns (64)-(65), (44)-(46) and (62) with Eqns (43), (47) and (53)-(54). The outer and inner radii $r_\pm$ and the radii $r_\pm^p$ of the elastic-plastic interfaces are given by

$$r_\pm/l_0 = \lambda_\theta^\pm/\alpha \;,\; r_\pm^p/l_0 = \lambda_\theta^{p\pm}/\alpha \quad (66)$$

6. INCOMPRESSIBLE DEFORMATION

For incompressible deformations ($\nu = 0.5$), Eqns (59)-(60) are reduced to

$$\lambda_\theta^{p+} = e^{(k/G)/2}/\sqrt{\lambda} = \lambda_{\theta 0}^+\big|_{\nu=0.5}$$

$$\lambda_\theta^{p-} = e^{-(k/G)/2}/\sqrt{\lambda} = \lambda_{\theta 0}^+\big|_{\nu=0.5} \quad (67)$$

Equation (61) remains unchanged. Functions Φ, $\Psi_\pm$ and $\Theta_\pm$ reduce to

$$\Phi = \Psi_\pm = \Theta_\pm = \eta_\pm = 1 \qquad (68)$$

The above results further lead to very much simplified expressions.

7. NUMERICAL EXAMPLES

As an example, a material with $\nu = 0.3$, $k/G = 0.01$ has been considered. The $2\alpha h_0/l_o - \lambda_\theta^+$ curve and the $M/h_0^2 - \lambda_\theta^+$ curve for this material for several values of the stretch λ are shown in Figs 3 and 4. Then, the two curves are combined into the $M/h_0^2 - 2\alpha h_0/l_0$ curve, shown in Fig. 5

For the same values of ν and k/G and for $\lambda = 1$ in Fig. 6, the development of the elastic zone versus the bending angle $2\alpha h_0/l_0$ is plotted. It is rapidly decreasing to form a narrow band[13] moving towards $-h_0$.(see, e.g., Ref 13)

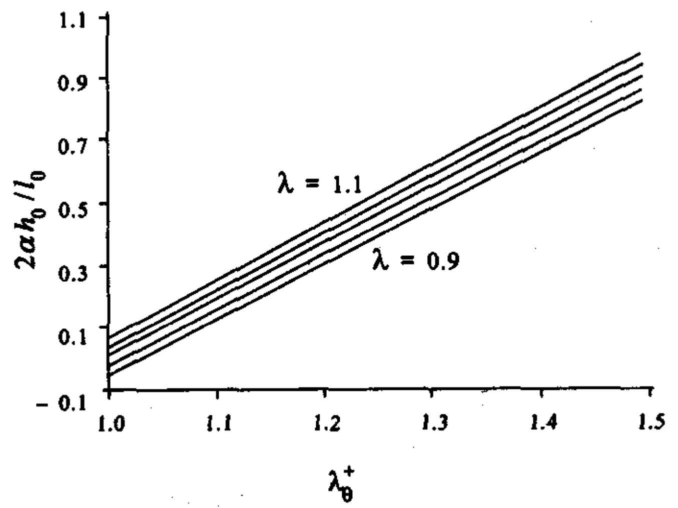

Figure 3. $2\alpha h_0/l_0$ versus λ_θ^+

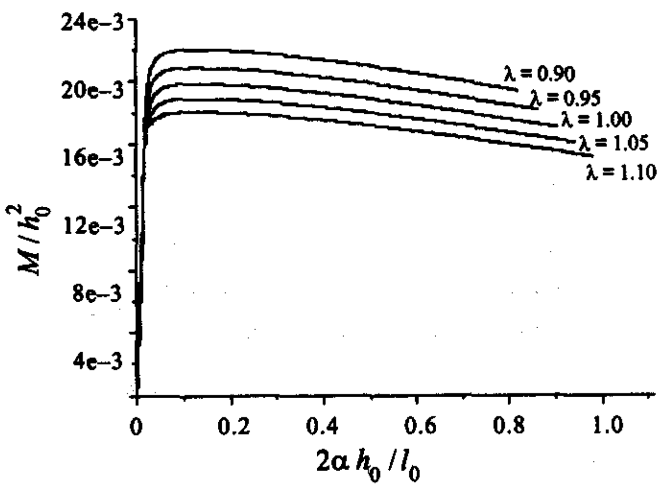

Figure 5. M/h_0^2 versus $2\alpha h_0/l_0$

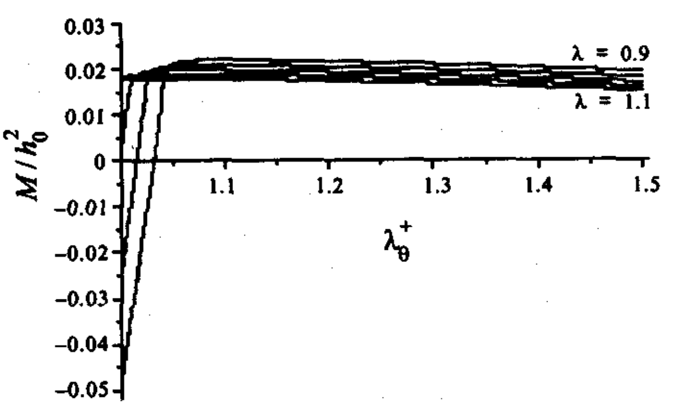

Figure 4. M/h_0^2 versus λ_θ^+

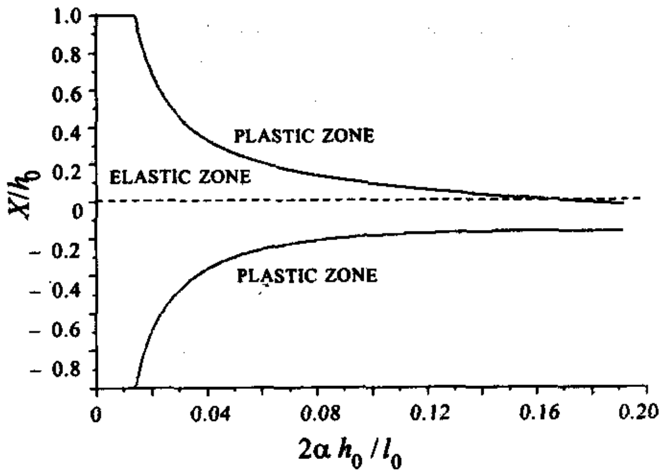

Figure 6. Development of the elastic zone as a fünctrion of $2\alpha h_0/l_0$.

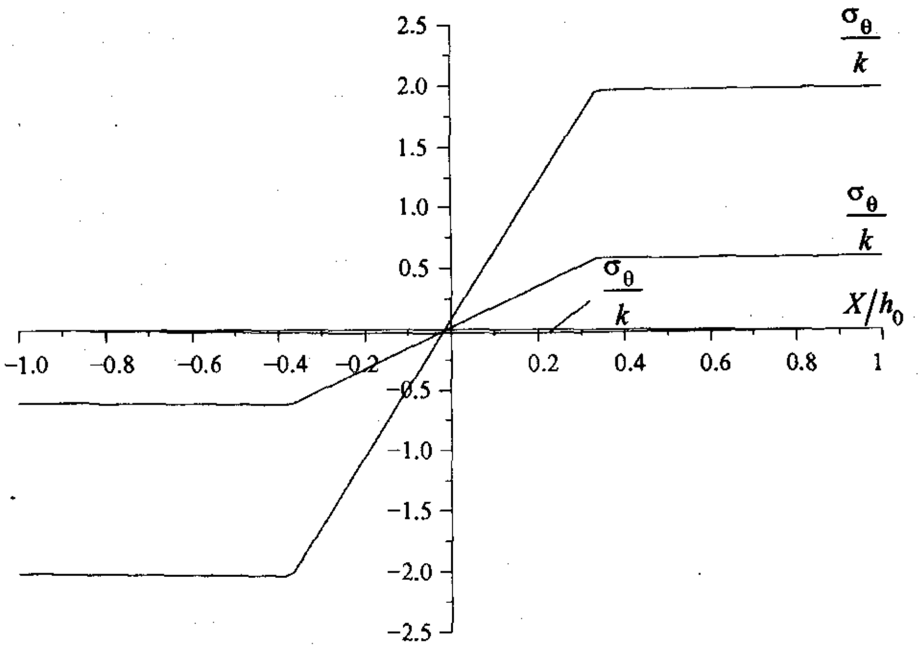

Figure 7. Principal stress distribution

Finally, the distribution of the principal stresses σ_r, σ_θ and σ_z on the current section θ = const. in the bent block are calculated, as shown by Fig. 7. Here λ_θ^+ has been chosen to be 1.2; the corresponding value for $2\alpha\, h_0/l_0$ is 0.0396154.

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