Defence Science Journal, Vol. 54, No. 2, April 2004, pp. 135-141

Elastic-plastic Transition of a Non-homogeneous Thick-walled Circular Cylinder under Internal Pressure

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ABSTRACT

Elastic-plastic stresses have been derived for a cylinder under internal pressure using Seth's transition theory. The effect of non-homogeneity has been discussed numerically and depicted graphically. It has been observed that a cylinder made of non-homogeneous material with non-homogeneity increasing radially requires a higher percentage increase in pressure to become fully plastic than to its initial yielding as compared to a homogeneous cylinder and is on the safer side of the design.

Keywords: Elastic-plastic transition, non-homogeneous materials, elastic, plastic, cylinder, elastic-plastic stresses, transition theory, non-homogeneity

NOMENCLATURE

e_{ii}^{A}	Principal finite strain components
a,b	Internal and external radii of cylinder
p	Internal pressure
u, v, z	Displacement components
r, θ, z	Radial, circumferential, and axial directions, respectively
e_{iji}, T_{ij}	Strain and stress tensors
δ_{ij}	Kronecker's delta
Y	Yield stress

1. INTRODUCTION

The constantly increasing industrial demand for axisymmetrical and spherical components or their elements has attracted the attention of designers and scientists on this particular area of activity. The progressive worldwide scarcity of materials combined with their consequently higher costs, makes it increasingly less attractive to confine design to the customary elastic regime only.

Thick-walled cylinders of circular cross sections are used commonly, either as pressure vessels intended for storage of industrial gases or as media for transportation of high pressurised fluids. Thick-walled cylinders under internal pressure have already been analysed by many authors[1-3] for isotropic homogeneous elastic-plastic states. Some degree of non-homogeneity is present in wide-class materials, such as hot-rolled copper, aluminum and magnesium alloys. Olszak and Urbanowski[7] have solved the problem of a thick-walled non-homogeneous cylinder subjected to internal and external pressures and showed that plastic flow may start from either surface, depending upon the characteristic and intensity of the non-

Revised 03 September 2002

homogeneity. They, however, assumed the material to be elastically incompressible.

In this paper, elastic-plastic stresses for non-homogeneous thick-walled cylinders subjected to internal pressure have been obtained using Seth's transition theory[5]. Seth[6] has defined the generalised principal strain measure as

$$e_{ii} = \int_0^{e_{ii}^A} \left[1-2e_{ii}^A\right]^{\frac{n}{2}-1} de_{ii}^A = (1/n)\left[1-\left(1-2e_{ii}^A\right)^{\frac{n}{2}}\right] \quad (i=1,2,3) \tag{1}$$

where n is the measure and e_{ii}^A are the principal finite components of strain.

Non-homogeneity is taken as the compressibility of the material in the cylinder as

$$C = C_0 r^{-k} \tag{2}$$

where $a \le r \le b$: C_0 and $k(\le 0)$ are the constants.

The results obtained have been discussed numerically and depicted graphically.

2. GOVERNING EQUATIONS

Consider a non-homogeneous thick-walled circular cylinder of internal and external radii a and b respectively, subjected to internal pressure, p. The non-homogeneity in the cylinder is due to variation in compressibility, C. In cylindrical polar coordinates, the displacements[6] are given by

$$u = r\,(1-\beta)\ ;\ v = 0\ ;\ w = dz \tag{3}$$

where β is a function of $r=\sqrt{x^2+y^2}$ and d is a constant. The finite components of strain are:

$$\left.\begin{aligned} e_{rr}^A &= (1/2)\left[1-(r\beta'+\beta)^2\right] \\ e_{\theta\theta}^A &= (1/2)\left[1-\beta^2\right] \\ e_{zz}^A &= (1/2)\left[1-(1-d)^2\right] \\ e_{r\theta}^A &= e_{\theta z}^A = e_{zr}^A = 0 \end{aligned}\right\} \tag{4}$$

Substituting Eqn (4) in Eqn (1), the generalised components of strain are:

$$\left.\begin{aligned} e_{rr} &= (1/n)\left[1-(r\beta'+\beta)^n\right] \\ e_{\theta\theta} &= (1/n)\left[1-\beta^n\right] \\ e_{zz} &= (1/2)\left[1-(1-d)^n\right] \\ e_{r\theta} &= e_{\theta z} = e_{zr} = 0 \end{aligned}\right\} \tag{5}$$

where $\beta' = d\beta/dr$.

The stress-strain relation[7] for elastic isotropic material is:

$$T_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} \qquad (i,j=1,2,3) \tag{6}$$

where λ and μ are Lame's constants, δ_{ij} is Kronecker's delta, and e_{kk} is the first strain invariant.

Substituting the strain components from Eqn (5) in Eqn (6), one gets:

$$\left.\begin{aligned} T_{rr} &= \lambda I_1 + (2\mu/n)\left[1-(r\beta'+\beta)^n\right] \\ T_{\theta\theta} &= \lambda I_1 + (2\mu/n)\left[1-\beta^n\right] \\ T_{zz} &= \lambda I_1 + (2\mu/n)\left[1-(1-d)^n\right] \\ T_{r\theta} &= T_{\theta z} = T_{zr} = 0 \end{aligned}\right\} \tag{7}$$

where $I_1 = (1/n)\left[3-(r\beta'+\beta)^n-\beta^n-(1-d)^n\right]$

All the equations of equilibrium are satisfied except

$$\frac{d}{dr}(T_{rr}) + \left[(T_{rr}-T_{\theta\theta})/r\right] = 0 \tag{8}$$

Using Eqn (7) in Eqn (8), one gets a nonlinear differential equation in β as

$$\begin{aligned} nP\beta(P+1)^{n-1}\frac{dP}{d\beta} &= r\left(\frac{\mu'}{\mu}-\frac{C'}{C}\right) \\ &\left[\left\{(3-2C)-(1-C)(1-d)^n\right\}\frac{1}{\beta^n}-(1-C)-(P+1)^n\right] \\ &+C\left[1-(P+1)^n\right]+rC'\left[1-\left\{2-(1-d)^n\right\}\frac{1}{\beta^n}\right] \\ &-nP\left[(1-C)+(P+1)^n\right] \end{aligned} \tag{9}$$

where $r\beta' = \beta P$, $C = 2\mu/(\lambda + 2\mu)$.

The transition points of β in Eqn (9), are $P \to -1$ and $P \to \pm\infty$

The boundary conditions are:

$$\begin{aligned} T_{rr} &= -p \quad \text{at} \quad r = a \\ T_{rr} &= 0 \quad \text{at} \quad r = b \end{aligned} \tag{10}$$

The resultant axial force in the cylinder is given by

$$L = 2\pi \int_a^b r T_{zz}\, dr \tag{11}$$

3. SOLUTION THROUGH PRINCIPAL STRESS

It has been shown that the asymptotic solution through the principal stress leads from elastic to plastic state at transition point $P \to \pm\infty$, one defines the transition function R_1 as

$$\begin{aligned} R_1 &= T_{rr} - \frac{\lambda}{n} k \\ &\equiv \frac{2\mu}{Cn}\left[C - \beta^n \left\{(1-C) + (P+1)^n\right\}\right] \end{aligned} \tag{12}$$

Taking the logarithmic differentiation of Eqn (12) wrt r, one gets:

$$\frac{d}{dr}\log R_1 = \frac{\begin{bmatrix} rC'(1+\beta^n) - nP\beta^{n+1}(P+1)^{n-1}\dfrac{dP}{d\beta} - \\ nP\beta^n\left\{(1-C)+(P+1)^n\right\} + \\ r\left(\dfrac{\mu'}{\mu} - \dfrac{C'}{C}\right)\left\{C - \beta^n\left[(1-C)+(P+1)^n\right]\right\} \end{bmatrix}}{r\left[C - \beta^n\left\{(1-C)+(P+1)^n\right\}\right]} \tag{13}$$

Substituting the value of $dP/d\beta$ from Eqn (9) in Eqn (13) and taking asymptotic value as $P \to \pm\infty$, one obtains:

$$(d/dr)\log R_1 = -C/r \tag{14}$$

Integrating Eqn (14), one gets:

$$R_1 = A\exp f(r) \tag{15}$$

where A is a constant of integration and

$$f(r) = -\int \frac{C}{r}\, dr \tag{16}$$

From Eqns (12) and (15), one has:

$$T_{rr} = A\exp f(r) + B \tag{17}$$

where $B = (\lambda/n)k$ and $k = [3 - (1-d)^n]$ (18)

Using boundary conditions [Eqn (10) in Eqn (17)], one gets:

$$A = \left(-P/\{\exp f(a) - f(b)\}\right); \quad B = -A\exp f(b) \tag{19}$$

From Eqns (5), (18), and (19), one has:

$$e_{zz} = \frac{\exp f(b)}{\lambda}\left[\frac{p}{\exp f(a) - \exp f(b)}\right] - \frac{2}{n} \tag{20}$$

Substituting the value of B from Eqn (19) in Eqn (17), one gets:

$$T_{rr} = A\left[\exp f(r) - \exp f(b)\right] \tag{21}$$

Using Eqn (21) in Eqn (8), one gets:

$$T_{\theta\theta} = A\left[(1-C)\exp f(r) - \exp f(b)\right] \tag{22}$$

Equation (7) reduces to

$$T_{zz} = \left(\frac{1-C}{2-C}\right)(T_{rr} + T_{\theta\theta}) + \frac{C\lambda}{(1-C)}\left(\frac{3-2C}{2-C}\right)e_{zz} \tag{23}$$

Using Eqn (20) and the closed-end condition [Eqn (11) in Eqn (23)], one gets the axial force L as

$$L = 2\pi\left(\frac{1-C(a)}{2-C(a)}\right)pa^2 + 2\pi\int_a^b \frac{r^2 C' T_{rr}}{(2-C)^2}dr + 2\pi\left\{\left[\frac{p\exp f(b)}{\exp f(a)-\exp f(b)}\right] - \frac{2\lambda}{n}\right\} \quad (24)$$

Equations (21)-(23) give the elastic-plastic transitional stresses for a non-homogeneous compressible cylinder under internal pressure.

Taking the non-homogeneity in the cylinder due to variable compressibility as given by Eqn (2), and substituting it in Eqns (21)-(24), one gets:

$$T_{rr} = A_1\left[\exp(C_0 r^{-k}/k) - \exp(C_0 b^{-k}/k)\right] \quad (25)$$

$$T_{\theta\theta} = T_{rr} - A_1\left[C_0 r^{-k}\exp(C_0 r^{-k}/k)\right] \quad (26)$$

$$T_{zz} = \left[\left(1-C_0 r^{-k}\right)/\left(2-C_0 r^{-k}\right)\right]\left(T_{rr}+T_{\theta\theta}\right) + A_2 e_{zz} \quad (27)$$

where

$$e_{zz} = \frac{\exp(C_0 b^{-k}/k)}{\lambda}\left[\frac{p}{\exp(C_0 a^{-k}/k) - \exp(C_0 b^{-k}/k)}\right] - \frac{2}{n} \quad (28)$$

and

$$A_1 = \frac{-p}{\exp(C_0 a^{-k}) - \exp(C_0 b^{-k})}$$

$$A_2 = \frac{\lambda C_0 r^{-k}(3-2C_0 r^{-k})}{(1-C_0 r^{-k})(2-C_0 r^{-k})}$$

From Eqns (25)-(26), one has:

$$T_{\theta\theta} - T_{rr} = A_1\left(-C_0 r^{-k}\exp(C_0 r^{-k}/k)\right) \quad (29)$$

It is seen from Eqn (29) that $|T_{\theta\theta} - T_{rr}|$ is maximum at $r=(-C_0/k)^{1/k}$; $k<0$. Therefore, yielding in the cylinder made of non-homogeneous material will take place at $r=(-C_0/k)^{1/k}=r_1$ (say) depending upon the values of $C_0 = C_1$ (say) and k.

For yielding at $r=(-C_1/k)^{1/k}=r_1$; $k<0$, Eqn (29) becomes:

$$\left|T_{\theta\theta} - T_{rr}\right|_{r=r_1} = \left|-A_1 C_0 r_1^{-k}\exp\left(C_0 r_1^{-k}/k\right)\right| \equiv Y \quad (30)$$

or $$\left|P_i A_6\right| \equiv 1 \quad \Rightarrow \quad P_i = \frac{1}{\left|A_6\right|} \quad (31)$$

where $P_i = \dfrac{p}{Y}$ and

$$A_6 = \frac{k}{e\left[\exp\left(\frac{C_1}{k}a^{-k}\right) - \exp\left(\frac{C_1}{k}b^{-k}\right)\right]}$$

From Eqn (30), the pressure required[8-9] for fully plastic state $C_0 \rightarrow 0$ is given by

$$(P/Y) = \left[C_1\left(a^{-k}-b^{-k}\right)/k^2\right] = P_f \quad (32)$$

The stresses [Eqns (25)-(27)] for fully plastic state become:

$$\left(T_{rr}/Y\right) = -P_f\left[\left\{(b/r)^k - 1\right\}/\left\{(b/a)^k - 1\right\}\right] \quad (33)$$

$$\left(T_{\theta\theta}/Y\right) = -P_f\left[\left\{\left((1-k)(b/r)^k\right) - 1\right\}/\left\{(b/a)^k - 1\right\}\right] \quad (34)$$

$$\left(T_{zz}/Y\right) = (1/2)\left[(T_{rr}/Y) + (T_{\theta\theta}/Y)\right] + \left[\left(3P_f k r^{-k}\right)/\left\{(2-k)(a^{-k}-b^{-k}\right\}\right] \quad (35)$$

and for the axial force given by Eqn (24) is:

$$(L/Y) = P_f \pi a^2 + \left[\left\{3\pi k P_f\left(b^{2-k} - a^{2-k}\right)\right\}/\left\{(2-k)\left(a^{-k}-b^{-k}\right)\right\}\right] \quad (36)$$

For homogeneous material ($k = 0$), the stresses [Eqns (33)–(35)] for fully plastic state become:

$$\left(T_{rr}/Y\right)=P_f\left(\log(r/b)\right)/\left(\log(b/a)\right) \tag{37}$$

$$\left(T_{\theta\theta}/Y\right)=P_f\left(1+\log(r/b)\right)/\left(\log(b/a)\right) \tag{38}$$

$$\left(T_{zz}/Y\right)=(1/2)\left[\left(T_{rr}/Y\right)+\left(T_{\theta\theta}/Y\right)\right]+\left[\left(3\{P_f-3\theta_1\}\right)/\left(2\{\log(b/a)\}\right)\right] \tag{39}$$

and the axial force given by Eqn (36) for $k = 0$ is:

$$(L/Y)=P_f\pi a^2+\left[\left\{3\pi(b^2-a^2)(P_f-3\theta_1)\right\}/\left\{2\log(b/a)\right\}\right] \tag{40}$$

Equations (37)-(39) are the same as obtained by Blazynski[1], Chakrabarty[2] and Hill[3].

4. NUMERICAL ILLUSTRATION & DISCUSSION

To observe the effects of pressure on a cylinder made of non-homogeneous material, Table 1 is given for radii ratios (b/a)= 4, 6, and 8. For a circular cylinder made of homogenous material ($k = 0$), yielding starts at internal surface whereas for a circular cylinder made of a non-homogeneous material ($k < 0$), non-homogeneity increases radially and yielding takes place at any radius r where $a < r < b$ at different temperatures, depending upon the values of C_0 and k (Table 1). It is seen from Table 1 that for a homogeneous cylinder, high pressure is required for initial yielding than for

Table 1. Values of pressure required for initial yielding, P_i and fully plastic state, P_f, for different radii ratios. For non-homogeneous case, yielding takes place at $r=\left(-C_0/k\right)^{\frac{1}{k}}$

Radii ratio	Constant k	Initial yielding at	Constant C_0	Initial yielding P_i	Fully plastic state P_f	P^* (%)
$\frac{b}{a}=4$	0	Internal surface	0.500	1.000	1.39	017.89
	−1	—	—	—	—	—
	−2	$r = 1.47$	0.924	0.860	3.46	100.58
	−3	$r = 2.04$	0.346	0.810	2.42	72.85
	−4	$r = 2.43$	0.115	0.660	1.84	66.97
$\frac{b}{a}=6$	0	Internal surface	0.500	1.200	1.79	22.13
	−1	—	—	—	—	—
	−2	$r = 2.21$	0.411	1.100	3.60	80.91
	−3	$r = 3.05$	0.103	0.870	2.45	67.81
	−4	$r = 3.64$	0.023	0.680	1.85	64.76
$\frac{b}{a}=8$	0	Internal surface	0.500	1.300	2.08	26.49
	−1	$r = 1.08$	0.924	0.720	4.88	160.45
	−2	$r = 2.94$	0.231	1.210	3.64	73.44
	−3	$r = 4.05$	0.043	0.890	2.46	66.25
	−4	$r = 4.85$	0.007	0.680	1.85	64.94

$^*P=\left[\sqrt{\frac{P_f}{P_i}}-1\right]\times 100$ is the percentage increase in pressure and P_i for initial yielding to fully plastic state, P_f.

that of a non-homogeneous cylinder and this pressure goes on increasing with the increase in radii ratio b/a.

It is also observed from Table 1 that for a homogeneous cylinder where yielding takes place at internal surface, it requires a less percentage increase in pressure to become fully plastic than to its initial yielding and this percentage goes on increasing with the increase in radii ratio b/a. For a non-homogeneous cylinder ($k<0$), a high percentage increase in pressure is required by the cylinder to become fully plastic as compared to a homogeneous cylinder.

In Fig. 1, curves have been drawn between the stresses and radii ratios r/b. It is seen that circumferential stress is maximum at the external surface for a non-homogeneous circular cylinder and non-homogeneity increases the tangential stress at the external surface significantly.

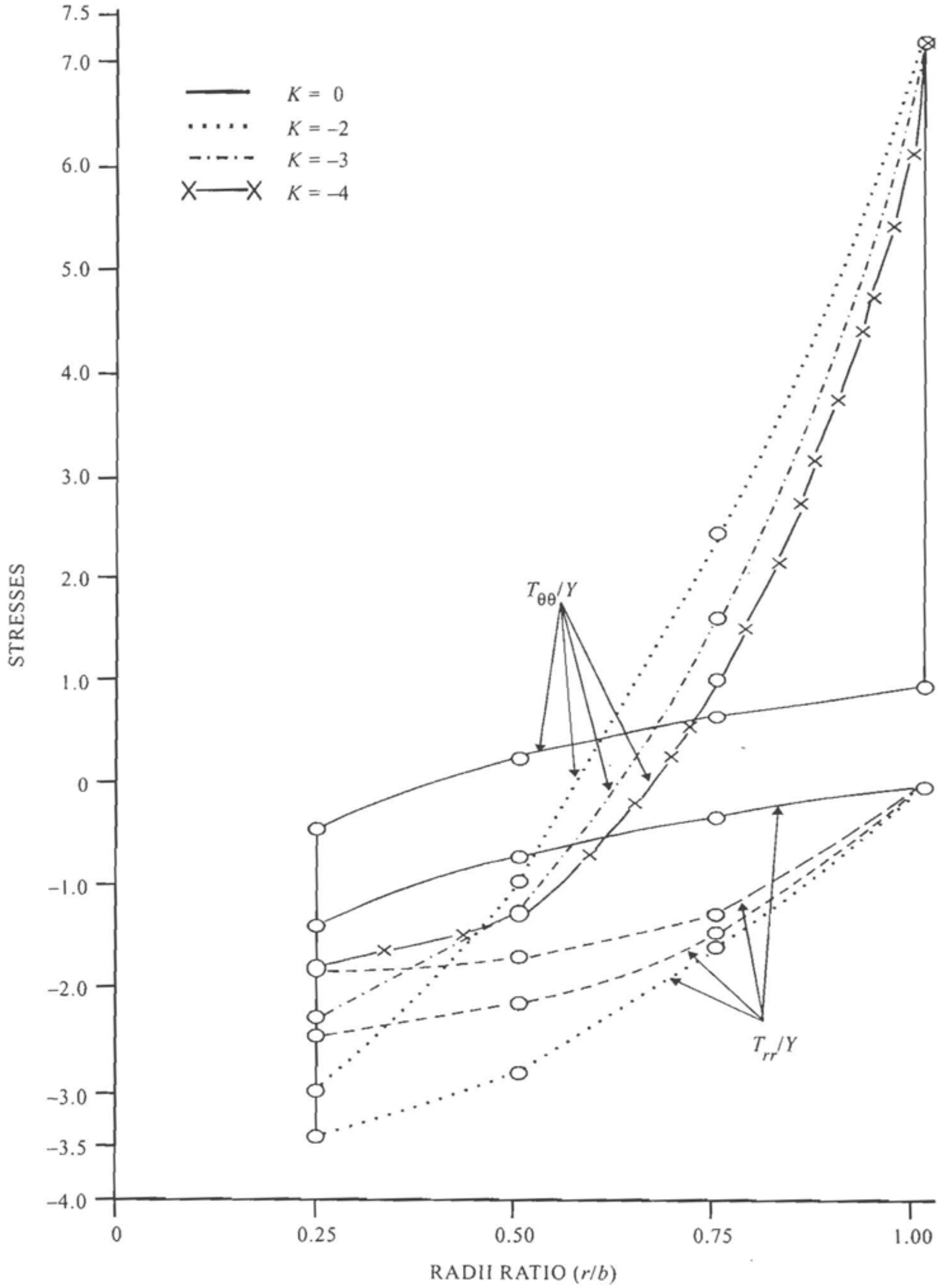

Figure 1. Plastic stress distribution in a non-homogeneous circular cylinder under internal pressure

CONCLUSION

It may be concluded that a circular cylinder made of non-homogeneous material (non-homogeneity increases radially) is on the safer side of design. Hence, the more use of non-homogeneous material (non-homogeneity increases radially), may therefore be beneficial for the manufacture of circular cylinders as they provide a longer service life than the cylinders made of homogeneous materials under identical conditions.

REFERENCE

1. Blazynski, T.Z. Applied elastico-plasticity of solids. MacMillian Press Ltd, London, 1983.
2. Chakravotry, J. Theory of plasticity. McGraw-Hill Book Co. Inc, New York, 1987.
3. Hill, R. Mathematical theory of plasticity. Clarendon Press, Oxford, 1950.
4. Olszak, W. & Urbanowski, W. *Arch. Mech. Stos.*, 1955, **3**, 315.
5. Seth, B.R. Transition theory of elastic-plastic deformation, creep and relaxation. *Nature*, 1962, **195**, 896-97.
6. Seth, B.R. Measyre concept in mechanics. *Int. J. Nonlinear Mech.*, 1966, **I**, 35-40.
7. Sokolinikoff, I. S. Mathematical theory of elasticity, Ed. 2. McGraw-Hill Book Co. Inc, New York, 1956.
8. Gupta, S. K. & Sharma, Sanjeev. Thermo elastic-plastic transition of non-homogeneous thick-walled circular cylinder under internal pressure. *Ind. J. Pure Appl. Math.*, 1997. **28** (12), 1621-633.
9. Gupta, S. K. & Sharma, Sanjeev. Thermo elastic-plastic transition in non-homogeneous thick-walled rotating cylinder. *In* 3rd International Congress on Thermal Stresses '99, June 13-17, 1999, Cracow, Poland.

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