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Magnetoelastic Torsional Vibration of Non-homogeneous Aeolotropic Cylindrical Shell of Viscoelastic Solids

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ABSTRACT

This study investigates magnetoelastic torsional vibration of a non-homogeneous aeolotropic cylindrical shell of viscoelastic solids. The non-homogeneity of the shell obeying power law variation of elastic constants and density given by $A_{ij} = C_{ij}r^n$, $\rho = \rho_0 r^n (i, j = 1,2,...6)$, where C_{ij} $(i, j = 1,2,...6)$ and ρ_0 are constants and r is the radius vector. Frequency equation and phase velocity in several cases have been derived. Such problems of interaction of elastic and electromagnetic fields have numerous applications in various branches of science, particularly in the detection of mechanical explosions in the interior of the earth and in the electromagnetic energy into vacuum.

Keywords: Magnetoelasticity, viscoelastic solids, magnetoelastic torsional vibration, torsional vibration, aeolotropic material

1. INTRODUCTION

Though the Maxwell equations governing the electromagnetic field have been known for quite a long time, the interest of researchers in the problems of interaction between elastic and magnetic fields has developed only a few decades ago. This is due to the possibilities of applying these coupled theories in practical situations, such as geophysics, optics, acoustics, damping of acoustic waves in magnetic fields, geomagnetics and oil prospecting, etc. For instance, Cagniard[1] while discussing the propagation of seismic waves from the earth's mantile to its inner core, suggested that earth's magnetic field may be considered for explaining certain phenomena concerning these waves. Knopoff[2] while working on this problem found the magnetic effect to be rather small. However, the theoretical developments in the subject continued with a view to find numerous applications in geographical and defence areas. Then, Kaliski[3] and Petykiewiez[4], Narain and Verma[5,6], Chandrasekhariah[7],Chakrabarti[8], Narain[9-11,13,14] and many others have investigated the magnetoelastic problems. Sequal to these, the present study investigates the magnetoelastic torsional vibration of a non-homogeneous cylindrical shell. Taking a special type of variation of elastic constants, the problem has been tried and solved in such a way that several

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published papers become a particular case of this study. Frequency equation in each case has been derived and the graphs have been plotted showing the effect of variation of elastic constants and the presence of magnetic field with initial stress.

2. FUNDAMENTAL EQUATIONS

Since the problem considered is of magneto-elasticity, the fundamental equations are those of elasticity and electromagnetism. The Maxwell equations governing the electromagnetic field are:

$$\text{curl } \vec{H} = 4\pi \vec{J}$$

$$\text{curl } \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{1}$$

$$\text{div } \vec{B} = 0$$

$$\vec{B} = \mu_e \vec{H}$$

where the displacement current is neglected and Gaussian units have been used.

Also, by Ohm's law

$$\vec{J} = \sigma\left(\vec{B} + \frac{1}{c}\frac{\partial \vec{u}}{\partial t} \times \vec{B}\right) \tag{2}$$

In Eqns (1) and (2), $\vec{H}$, $\vec{B}$, $\vec{E}$, $\vec{J}$ denote the magnetic intensity, magnetic induction, electric intensity, and current density vectors, respectively, μ_e and σ denote the magnetic permeability and the electric conductivity of the body, $\vec{u}$ represents displacement vector in the strained state and c the speed of light.

The electromagnetic field equations in vacuum are:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}^* = 0 \tag{3}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{h}^* = 0 \tag{4}$$

$$\text{curl } \vec{E}^* = -\frac{1}{c}\frac{\partial \vec{h}^*}{\partial t} \tag{5}$$

$$\text{curl } \vec{h}^* = -\frac{1}{c}\frac{\partial \vec{E}^*}{\partial t} \tag{6}$$

where $\vec{h}$ is the perturbation in magnetic field, and quantities with asterisk (*) represent the value of the corresponding quantities in vacuum. In a cylindrical coordinate system, if the body is under initial stress, $\bar{\sigma}_{33}$ along z-direction only, the stress equation of motion satisfied by incremental stresses and σ_{ij} initial stresses[10] are:

$$\frac{\partial}{\partial r}\sigma_{rr} + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{rz} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \bar{\sigma}_{33}\frac{\partial \omega_\theta}{\partial z} + F_r = \rho\frac{\partial^2 u_r}{\partial t^2} \tag{7}$$

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} + \frac{2}{r}\sigma_{r\theta} + \bar{\sigma}_{33}\frac{\partial \omega_r}{\partial z} + F_\theta = \rho\frac{\partial^2 u_\theta}{\partial t^2} \tag{8}$$

$$\frac{\partial}{\partial r}\sigma_{rz} + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta z} + \frac{\partial}{\partial z}\sigma_{zz} + \frac{1}{r}\sigma_{rz} + \bar{\sigma}_{33}\left(\frac{\partial \omega_\theta}{\partial r} - \frac{1}{r}\frac{\partial \omega_r}{\partial \theta}\right) + F_z = \rho\frac{\partial^2 u_z}{\partial t^2} \tag{9}$$

where σ_{ij} are the incremental stress components; ω_r, ω_θ are rotational components of strain, and $\bar{\sigma}_{33}$ is the initial stress along original z-direction. The stress-strain relations in cylindrical coordinates for aeolotropic elastic material as given by Love[15] are:

$$\sigma_{rr} = A_{11}^0 e_{rr} + A_{12}^0 e_{\theta\theta} + A_{13}^0 e_{zz}$$

$$\sigma_{\theta\theta} = A_{21}^0 e_{rr} + A_{22}^0 e_{\theta\theta} + A_{23}^0 e_{zz}$$

$$\sigma_{zz} = A_{31}^0 e_{rr} + A_{32}^0 e_{\theta\theta} + A_{33}^0 e_{zz}$$

$$\sigma_{rz} = A_{44}^0 e_{rz}$$

$$\sigma_{\theta z} = A_{55}^0 e_{\theta z}$$

$$\sigma_{r\theta} = A_{66}^0 e_{r\theta} \tag{10}$$

where $A_{ij}(i, j = 1, 2, 3, 4, 5, 6)$ are the elastic constants and $e_{rr}, e_{\theta\theta}$..........etc. are the strain components given by

$$e_{rr} = \frac{\partial u_r}{\partial r}, e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{u_r}{r}, e_{zz} = \frac{\partial u_z}{\partial z}$$

$$2e_{\theta z} = \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial u_\theta}{\partial z}, 2e_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$$

$$2e_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_r}{r} + \frac{2}{r}\frac{\partial u_r}{\partial \theta} \tag{11}$$

where $\vec{u} = [u_r, u_\theta, u_z]$ and the rotational components are given by

$$\omega_r = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)$$

$$\omega_\theta = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)$$

$$\omega_z = \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right) \tag{12}$$

Also, $\vec{F} = [F_r, F_\theta, F_z]$ is the Lorentz force per unit volume due to axial magnetic field and is given by

$$\vec{F} = \vec{J} \times \vec{B} \tag{13}$$

3. CONSIDERATION OF PROBLEM & METHOD OF SOLUTION

Let a semi-infinite cylindrical shell of radii a and b under initial stress be considered which has been taken as tension P along the axis of the shell. Suppose that elastic properties of the shell are symmetrical about z-axis, and the shell is placed in an axial magnetic field surrounded by vacuum. Since the torsional vibration of an aeolotropic cylindrical shell is being investigated therefore, the displacement vector $\vec{u}$ has only v as its non-vanishing component which is independent of θ. Thus, the displacement vector

$$u_r = 0, \; u_z = 0, \; u_\theta = v(r,z) \tag{14}$$

and the strain components take the form:

$$e_{rr} = e_{\theta\theta} = e_{zr} = 0$$

$$2e_{\theta z} = -\frac{\partial v}{\partial z}, 2e_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r}$$

$$2\omega_r = -\frac{\partial v}{\partial z}, 2\omega_z = \frac{\partial v}{\partial r} + \frac{v}{r}, \omega_\theta = 0 \tag{15}$$

It has been assumed that the elastic constants of medium exhibiting the specified property are given by

$$A_{ij}^0 = A_{ij} + A_{ij}' \frac{\partial}{\partial t} + A_{ij}'' \frac{\partial^2}{\partial t^2} \tag{16}$$

Using Eqns (14) and (15), Eqn (10) becomes:

$$\sigma_{r\theta} = \sigma_{\theta\theta} = \sigma_{rz} = 0$$

$$\sigma_{r\theta} = \left(A_{66} + A_{66}' \frac{\partial}{\partial t} A_{66}'' \frac{\partial^2}{\partial t^2}\right)\frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)$$

$$\sigma_{\theta z} = \left(A_{55} + A_{55}' \frac{\partial}{\partial t} + A_{55}'' \frac{\partial^2}{\partial t^2}\right)\left(-\frac{1}{2}\frac{\partial v}{\partial r}\right) \tag{17}$$

Let one supposes that the magnetic intensity vector $\vec{H}$ has the components $H_r = H_\theta = 0$ and $H_z = H$ (constant).

Also

$$\vec{H} = \vec{H}_0 + \vec{h} \tag{18}$$

where $\vec{H}_0$ is the initial magnetic field acting along z-axis and $\vec{h}$ is the perturbation in the field.

If the shell is considered as a perfect conductor of electricity, ie, $\sigma \to \infty$, then the Eqn (2) takes the form:

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{u}}{\partial t} \times \vec{B}$$

or

$$\vec{E} = \left[-\frac{\mu_e H}{c}\frac{\partial v}{\partial t}, 0, 0 \right] \quad (19)$$

Using the Eqns (1) and (19), the Eqn (18) becomes:

$$\vec{h} = \left[0, H\frac{\partial v}{\partial z}, 0 \right] \quad (20)$$

The Eqns (1) and (20) take the form:

$$\vec{F} = \left[0, -\frac{H^2}{4\pi}\frac{\partial^2 v}{\partial z^2}, 0 \right] \quad (21)$$

In view of the Eqns (19) and (20), one gets:

$$\vec{E} = \left[E^*, 0, 0 \right] \text{ and } \vec{h}^* = \left[0, h^*, 0 \right] \quad (22)$$

Since the shell is initially stressed along z-axis, the initial stress is taken as $\bar{\sigma}_{33} = -P/2$. Using the Eqns (17) and (21), it was found that the Eqns (7) and (9) are identically satisfied, and the remaining Eqn (8) gives:

$$\frac{\partial}{\partial r}\left(A_{66} + A'_{66}\frac{\partial}{\partial t} + A''_{66}\frac{\partial^2}{\partial t^2} \right)\frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} \right)$$
$$+ \frac{\partial}{\partial z}\left[\left(A_{55} + A'_{55}\frac{\partial}{\partial t} + A''_{55}\frac{\partial^2}{\partial t^2} \right)\frac{1}{2}\frac{\partial v}{\partial z} \right]$$
$$+ \frac{2}{r}\left(A_{66} + A'_{66}\frac{\partial}{\partial t} + A''_{66}\frac{\partial^2}{\partial t^2} \right)\frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} \right)$$
$$-\left(\frac{H^2}{4\pi} + \frac{P}{2} \right)\frac{\partial^2 v}{\partial z^2} = \rho\frac{\partial^2 v}{\partial t^2} \quad (23)$$

Suppose

$$A_{ij} = C_{ij}r^n, \; A'_{ij} = C'_{ij}r^n, \; A''_{ij} = C''_{ij}r^n \text{ and } \rho = \rho_0 r^n \quad (24)$$

where C_{ij}, C'_{ij}, C''_{ij} and ρ_0 are constants and n is any integer, then the Eqn (17) takes the form:

$$\sigma_{\theta z} = \left(C_{55} + C'_{55}\frac{\partial}{\partial t} + C''_{55}\frac{\partial^2}{\partial t^2} \right) r^n \frac{1}{2}\frac{\partial v}{\partial z}$$

$$\sigma_{r\theta} = \left(C_{66} + C'_{66}\frac{\partial}{\partial t} + C''_{66}\frac{\partial^2}{\partial t^2} \right) r^n \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \quad (25)$$

As a consequence of the Eqn (24), the Eqn (23) becomes:

$$\frac{\partial}{\partial r}\left[\left(C_{66} + C'_{66}\frac{\partial}{\partial t} + C''_{66}\frac{\partial^2}{\partial t^2} \right) r^n \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \right]$$
$$+ \frac{\partial}{\partial z}\left[\left(C_{55} + C'_{55}\frac{\partial}{\partial t} + C''_{55}\frac{\partial^2}{\partial t^2} \right) r^n \frac{1}{2}\frac{\partial v}{\partial z} \right]$$
$$+ \frac{2}{r}\left[\left(C_{66} + C'_{66}\frac{\partial}{\partial t} + C''_{66}\frac{\partial^2}{\partial t^2} \right) r^n \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \right]$$
$$-\left(\frac{H^2}{4\pi} + \frac{P}{2} \right)\frac{\partial^2 v}{\partial z^2} = \rho_0 r^n \frac{\partial^2 v}{\partial t^2} \quad (26)$$

Suppose

$$v = V(r)e^{i(\alpha z + pt)}$$

then the Eqn (26) becomes:

$$\frac{\partial^2 V}{\partial r^2} + \frac{(n+1)}{r}\frac{\partial V}{\partial r} - \frac{(n+1)}{r^2}V + K_1^2 V + K_2^2 \frac{V}{r^n} = 0 \quad (27)$$

where

$$K_1^2 = \frac{2\rho_0 p^2 - \left(C_{55} + C'_{55}ip - C''_{55}p^2 \right)\alpha^2}{\left(C_{66} + C'_{66}ip - C''_{66}p^2 \right)} \quad (28)$$

$$K_2^2 = \frac{\left(P + \frac{H^2}{2\pi}\right)\alpha^2}{\left(C_{66} + C'_{66}ip - C''_{66}p^2\right)} \tag{29}$$

As the Eqn (27) becomes very cumbersome, its solution is obtained for some particular values of n, say $n = 0, 2$.

4. SPECIAL CASES

Case I

Let $n = 0$, in this case, the Eqn (27) becomes:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \left(M^2 - \frac{1}{r^2}\right)V = 0 \tag{30}$$

where

$$M^2 = K_1^2 + K_2^2 \tag{31}$$

The solution of the Eqn (30) is given by

$$V = AJ_1(Mr) + BY_1(Mr)$$

which gives:

$$v = \left[AJ_1(Mr) + BY_1(Mr)\right]e^{i(\alpha z + pt)} \tag{32}$$

Now, the Eqn (25) with the help of the Eqn (32) becomes:

$$\sigma_{r\theta} = e^{i(\alpha z + pt)}\left(C_{66} + C'_{66}ip - C''_{66}p^2\right) \begin{bmatrix} \frac{A}{2}\left\{MJ_0(Mr) - \frac{2}{r}J_1(Mr)\right\} \\ + \frac{B}{2}\left\{MY_0(Mr) - \frac{2}{r}Y_1(Mr)\right\} \end{bmatrix} \tag{33}$$

The boundary conditions which must be satisfied are:

$$\begin{cases} \sigma_{r\theta} + T_{r\theta} = T^*_{r\theta} \text{ on } r = a \\ \sigma_{r\theta} + T_{r\theta} = T^*_{r\theta} \text{ on } r = b \end{cases} \tag{34}$$

where $T_{r\theta}$ and $T^*_{r\theta}$ are the Maxwell stresses in the body and in the vacuum, respectively. It can be easily verified[8] that

$$T_{r\theta} = T^*_{r\theta} = 0 \tag{35}$$

The Eqn (19) with the help of Eqn (32) takes the form:

$$E = -\frac{\mu_e H}{c} ip\{AJ_1(Mr) + BY_1(Mr)\}e^{i(\alpha z + pt)} \tag{36}$$

Suppose

$$E^* = E_0^* e^{i(\alpha z + pt)} \tag{37}$$

then the Eqn (3) becomes:

$$\frac{\partial E_0^*}{\partial r^2} + \frac{1}{r}\frac{\partial E_0^*}{\partial r} + \beta^2 E_0^* = 0 \tag{38}$$

where

$$\beta^2 = \frac{p^2}{c^2} - \alpha^2 \tag{39}$$

The solution of the Eqn (38) is given by

$$E_0^* = CJ_0(\beta r) + DY_0(\beta r)$$

where J_0 and Y_0 are the Bessel functions of order zero; C and D are constants. As a consequence of the Eqn (37) this solution becomes:

$$E^* = \{CJ_0(\beta r) + DY_0(\beta r)\}e^{i(\alpha z + pt)} \tag{40}$$

The boundary conditions [Eqn (34)] with the help of the Eqns (33) and (35) turn into:

$$A\{MaJ_0(Ma) - 2J_1(Ma)\} + B\{MaY_0(Ma) - 2Y_1(Ma)\} = 0 \tag{41}$$

$$A\{MbJ_0(Mb) - 2J_1(Mb)\} + B\{MbY_0(Mb) - 2Y_1(Mb)\} = 0 \tag{42}$$

Eliminating A and B from the Eqns (41) and (42), one gets the frequency equation as

$$\begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{vmatrix} = 0 \tag{43}$$

where

$$X_{11} = MaJ_0(Ma) - 2J_1(Ma)$$

$$X_{12} = MaY_0(Ma) - 2Y_1(Ma)$$

$$X_{21} = MbJ_0(Mb) - 2J_1(Mb)$$

$$X_{22} = MbY_0(Mb) - 2Y_1(Mb)$$

Solving the Eqn (43), one gets the frequency equation as

$$\frac{MaJ_0(Ma) - 2J_1(Ma)}{MbJ_0(Mb) - 2J_1(Mb)} = \frac{MaY_0(Ma) - 2Y_1(Ma)}{MbY_0(Mb) - 2Y_1(Mb)} \tag{44}$$

If the shell is very thin, one put $b = a + \delta a$ and neglecting δa^2, δa^3.......,etc and using the following results of Watson[16]:

$$Y_\upsilon'(z) = \frac{\upsilon}{z} Y_\upsilon(z) - Y_{\upsilon+1}(z) \tag{45}$$

and

$$J_\upsilon'(z) = \frac{\upsilon}{z} J_\upsilon(z) - J_{\upsilon+1}(z) \tag{46}$$

one gets the frequency equation as

$$M^3 a^2 + M - 1 = 0 \tag{47}$$

where

$$M^2 = \frac{2\rho_0 p^2 - \left(C_{55} + C_{55}'ip - C_{55}''p^2\right)\alpha^2 + \left(P + \frac{H^2}{2\pi}\right)\alpha^2}{\left(C_{66} + C_{66}'ip - C_{66}''p^2\right)} \tag{48}$$

Putting the value of M from the Eqns (47) and (48), one finds the relation for frequency p of the torsional vibration. This relation contains both the terms P and H which show that the frequency is affected by the initial stress and the magnetic field. Setting $Ma = \xi$, the phase velocity $c_1 (= p/\alpha)$ may be written in the following form:

$$\frac{c_1^2}{\beta^2} = \xi^2 \left(\frac{\lambda}{2\pi a}\right)^2 + G - \frac{\left(P + \frac{H^2}{4\pi}\right)}{\left(C_{66} + C_{66}'ip - C_{66}''p^2\right)} \tag{49}$$

where λ, the wavelength $= \dfrac{2\pi}{\alpha}$

and

$$G = \frac{C_{55} + C_{55}'ip - C_{55}''p^2}{C_{66} + C_{66}'ip - C_{66}''p^2}$$

$$\beta^2 = \frac{C_{66} + C_{66}'ip - C_{66}''p^2}{2\rho_0}$$

From Eqn (49) it is clear that $\left(P + \frac{H^2}{4\pi}\right)$ is negative on the RHS which indicates that both the initial stress and the magnetic field reduce the phase velocity of the above type of vibration.

Case I(a)

For a cylindrical shell of an aeolotropic material

$$C_{ij}' = C_{ij}'' = 0$$

and one gets the frequency equation as

$$\xi_0^3 + \xi_0 - a = 0$$

Therefore, in this case, the phase velocity takes the form:

$$c_0^2 = \frac{C_{66}}{2\rho_0} \left\{ \xi_0^2 \left\{\frac{\lambda}{2\pi a}\right\}^2 + \frac{C_{55}}{C_{66}} - \frac{\left(P + \frac{H^2}{2\pi}\right)}{C_{66}} \right\}$$

or

$$\frac{c_0}{\beta}=\left\{\frac{\left(\frac{\xi_0}{2\pi}\right)^2}{\left(\frac{a}{\lambda}\right)^2}+\frac{C_{55}}{C_{66}}-\frac{\left(P+\frac{H^2}{2\pi}\right)}{C_{66}}\right\}^{\frac{1}{2}} \tag{50}$$

A negative sign before the term $\left(P+\frac{H^2}{2\pi}\right)\Big/C_{66}$ indicates that both the initial stress and the magnetic field reduce the phase velocity c_0, where

$$\beta^2 = C_{66}/2\rho_0$$

Case I(b)

For an isotropic cylindrical shell

$$C'_{ij}=C''_{ij}=0 \text{ and } C_{55}=C_{66}=\mu$$

Therefore, in this case, the phase velocity takes the form:

$$c_0^2=\frac{\mu}{2\rho_0}\left\{\xi_0^2\left(\frac{\lambda}{2\pi a}\right)^2+1-\frac{1}{\mu}\left(P+\frac{H^2}{2\pi}\right)\right\} \tag{51}$$

which is coincident with the result of Narain[10].

Case I(c)

If there was no initial stress, then $P = 0$ and the result of phase velocity for an isotropic cylindrical shell takes the form:

$$c_0^2=\frac{\mu}{2\rho_0}\left\{\xi_0^2\left(\frac{\lambda}{2\pi a}\right)^2+1-\left(\frac{H^2}{2\pi\mu}\right)\right\} \tag{52}$$

which is the same as obtained by Chandrasekharaiah[7].

Case II

Let $n = 2$, in this case, the Eqn (27) becomes:

$$\frac{\partial^2 V}{\partial r^2}+\frac{3}{r}\frac{\partial V}{\partial r}+\left\{K_1^2-\frac{\left(3-K_2^2\right)}{r^2}\right\}V=0 \tag{53}$$

Putting $V=\frac{1}{r}\psi(r)$ in the Eqn (53), one gets:

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial t}+\left\{K_1^2-\frac{\upsilon^2}{r^2}\right\}\psi=0 \tag{54}$$

where

$$\upsilon^2=3-K_2^2$$

The solution of the Eqn (54) becomes:

$$\psi=AJ_\upsilon(K_1r)+BY_\upsilon(K_1r)$$

Putting the value of ψ and V, one gets:

$$v=\frac{1}{r}\left[AJ_\upsilon(K_1r)+BY_\upsilon(K_1r)\right]e^{i(\alpha z+pt)} \tag{55}$$

From the Eqns (25) and (55), one gets:

$$\sigma_{r\theta}=e^{i(\alpha z+pt)}\left(C_{66}+C'_{66}ip-C''_{66}p^2\right)$$

$$\begin{bmatrix}\frac{A}{2}\{K_1rJ_{\upsilon-1}(K_1r)-(\upsilon+2)J_\upsilon(K_1r)\}\\+\frac{B}{2}\{K_1rY_{\upsilon-1}(K_1r)-(\upsilon+2)Y_\upsilon(K_1r)\}\end{bmatrix}=0 \tag{56}$$

Using the Eqns (35) and (56), the boundary conditions in the Eqn (34) reduce to:

$$\begin{bmatrix}\frac{A}{2}\{K_1aJ_{\upsilon-1}(K_1a)-(\upsilon+2)J_\upsilon(K_1a)\}\\+\frac{B}{2}\{K_1aY_{\upsilon-1}(K_1a)-(\upsilon+2)Y_\upsilon(K_1a)\}\end{bmatrix}=0 \tag{57}$$

$$\begin{bmatrix}\frac{A}{2}\{K_1bJ_{\upsilon-1}(K_1b)-(\upsilon+2)Y_\upsilon(K_1b)\}\\+\frac{B}{2}\{K_1bY_{\upsilon-1}(K_1b)-(\upsilon+2)Y_\upsilon(K_1b)\}\end{bmatrix}=0 \tag{58}$$

Eliminating A and B from the Eqns (57) and (58), one gets:

$$\begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix} = 0 \qquad (59)$$

where

$$Z_{11} = \{K_1 a J_{\upsilon-1}(K_1 a) - (\upsilon+2) J_\upsilon(K_1 a)\}$$

$$Z_{12} = \{K_1 a Y_{\upsilon-1}(K_1 a) - (\upsilon+2) Y_\upsilon(K_1 a)\}$$

$$Z_{21} = \{K_1 b J_{\upsilon-1}(K_1 b) - (\upsilon+2) J_\upsilon(K_1 b)\}$$

$$Z_{22} = \{K_1 b Y_{\upsilon-1}(K_1 b) - (\upsilon+2) Y_\upsilon(K_1 b)\}$$

Solving the determinant in Eqn (59), one gets the frequency equation as

$$\frac{\{K_1 a J_{\upsilon-1}(K_1 a) - (\upsilon+2) J_\upsilon(K_1 a)\}}{\{K_1 a Y_{\upsilon-1}(K_1 a) - (\upsilon+2) Y_\upsilon(K_1 a)\}} = \frac{\{K_1 b J_{\upsilon-1}(K_1 b) - (\upsilon+2) J_\upsilon(K_1 b)\}}{\{K_1 b Y_{\upsilon-1}(K_1 b) - (\upsilon+2) Y_\upsilon(K_1 b)\}} \qquad (60)$$

For a thin shell, with the same assumption as in the previous case, one gets the frequency equation as

$$(\upsilon+2)^2 - \left(2\upsilon - 1 + \frac{1}{K_1}\right)(\upsilon+2) + K_1^2 a^2 = 0 \qquad (61)$$

where

$$\upsilon^2 = 3 - K_2^2 = 3 - \frac{\left(P + \frac{H^2}{2\pi}\right)\alpha^2}{\left(C_{66} + C'_{66} ip - C''_{66} p^2\right)} \qquad (62)$$

$$K_1^2 = \frac{2\rho_0^2 p^2 - \left(C_{55} + C'_{55} ip - C''_{55} p^2\right)\alpha^2}{\left(C_{66} + C'_{66} ip - C''_{66} p^2\right)} \qquad (63)$$

Putting the value of υ from the Eqn (61) in Eqn (62), one finds the value of the frequency p. Since this equation contains both the terms P and H, the frequency of the wave generated due to such torsional vibration is affected by the initial stress and the magnetic field. In this case too, the Eqn (63) gives the phase velocity c of the torsional vibration as

$$\frac{c^2}{\beta^2} = \frac{\xi_1^2 \lambda^2}{(2\pi a)^2} + \left(\frac{C_{55} + C'_{55} ip - C''_{55} p^2}{C_{66} + C'_{66} ip - C''_{66} p^2}\right) \qquad (64)$$

where $\lambda = 2\pi/\alpha$ is the wavelength and ξ_1 is the root of the equation.

$$\frac{\{\xi_1 J_{\upsilon-1}(\xi_1) - (\upsilon+2) J_\upsilon(\xi_1)\}}{\{\xi_1 Y_{\upsilon-1}(\xi_1) - (\upsilon+2) Y_\upsilon(\xi_1)\}} = \frac{\{\xi_1 R_1 J_{\upsilon-1}(\xi_1 R_1) - (\upsilon+2) J_\upsilon(\xi_1 R_1)\}}{\{\xi_1 R_1 Y_{\upsilon-1}(\xi_1 R_1) - (\upsilon+2) Y_\upsilon(\xi_1 R_1)\}} \qquad (65)$$

with

$$R_1 = \frac{b}{a}$$

and

$$\beta^2 = \frac{\left(C_{66} + C'_{66} ip - C''_{66} p^2\right)}{2\rho_0} \qquad (66)$$

Case II(a)

For an aeolotropic cylindrical shell, one put:

$$C'_{ij} = C''_{ij} = 0$$

and gets frequency equation as

$$\frac{\{K_3 a J_{\upsilon_1-1}(K_3 a) - (\upsilon+2) J_{\upsilon_1}(K_3 a)\}}{\{K_3 a Y_{\upsilon_1-1}(K_3 a) - (\upsilon+2) Y_{\upsilon_1}(K_3 a)\}} = \frac{\{K_3 b J_{\upsilon_1-1}(K_3 b) - (\upsilon+2) J_{\upsilon_1}(K_3 b)\}}{\{K_3 b Y_{\upsilon_1-1}(K_3 b) - (\upsilon+2) Y_{\upsilon_1}(K_3 b)\}} \qquad (67)$$

or

$$\xi_2^3 + 6\xi_2 - 3a = 0$$

where

$$\upsilon_1^2 = 3 - \frac{\left(P + \frac{H^2}{2\pi}\right)\alpha^2}{C_{66}}$$

and

$$K_3^2 = \frac{2\rho_0 p^2 - C_{55}\alpha^2}{C_{66}}, \xi_2 = K_3 a \quad \text{at} \quad \upsilon_1 = 1$$

In this case, the phase velocity c_1 is given by

$$\frac{c_1^2}{\beta_1^2} = \frac{\xi_2^2 \lambda^2}{(2\pi a)^2} + \frac{C_{55}}{C_{66}}$$

or

$$\frac{c_1}{\beta_1} = \left\{ \frac{\left(\frac{\xi_2}{2\pi}\right)^2}{\left(\frac{a}{\lambda}\right)^2} + \frac{C_{55}}{C_{66}} \right\}^{\frac{1}{2}} \tag{68}$$

where

$$\beta_1^2 = \frac{C_{66}}{2\rho_0}$$

Case II(b)

For an isotropic cylindrical shell, one puts:

$$C'_{ij} = C''_{ij} = 0 \quad \text{and} \quad C_{55} = C_{66} = \mu .$$

The frequency equation for isotropic cylindrical shell takes the form:

$$\frac{\{K_4 a J_{\upsilon_2 - 1}(K_4 a) - (\upsilon + 2) J_{\upsilon_2}(K_4 a)\}}{\{K_4 a Y_{\upsilon_2 - 1}(K_4 a) - (\upsilon + 2) Y_{\upsilon_2}(K_4 a)\}} = \frac{\{K_4 b J_{\upsilon_2 - 1}(K_4 b) - (\upsilon + 2) J_{\upsilon_2}(K_4 b)\}}{\{K_4 b Y_{\upsilon_2 - 1}(K_4 b) - (\upsilon + 2) Y_{\upsilon_2}(K_4 b)\}} \tag{69}$$

where

$$\upsilon_2^2 = 3 - \frac{\left(P + \frac{H^2}{2\pi}\right)\alpha^2}{\mu}$$

and

$$K_4^2 = \frac{2\rho_0 p^2 - \mu\alpha^2}{\mu}$$

In this case, the phase velocity c_2 is given by

$$\frac{c_2^2}{\beta_2^2} = \frac{\xi_2^2 \lambda^2}{(2\pi a)^2} + 1 \tag{70}$$

where

$$\beta_2^2 = \frac{\mu}{2\rho_0}$$

5. NUMERICAL RESULTS & DISCUSSION

To have some idea about the effect of non-homogeneity on torsional vibration of the cylindrical shell of aeolotropic material, the following two cases have been considered:

Case A

Homogeneous shell, ie, when $n = 0$. In this case, the Eqn (50) is considered and $\xi_0 = 2.332$, $a = 15$, $\frac{C_{55}}{C_{66}} = 0.9$, $\frac{\left(P + \frac{H^2}{4\pi}\right)}{C_{66}} = 0.4$ are taken and values of (c_0/β) for different values of (a/λ) are depicted in Table 1 (*Case A*) and a graph *A* is plotted between these (Fig. 1).

Case B

When both the initial stress and the magnetic field are absent, but the elastic constants and the density of the material of the shell are varying as the square of the radius vector. In this case, the

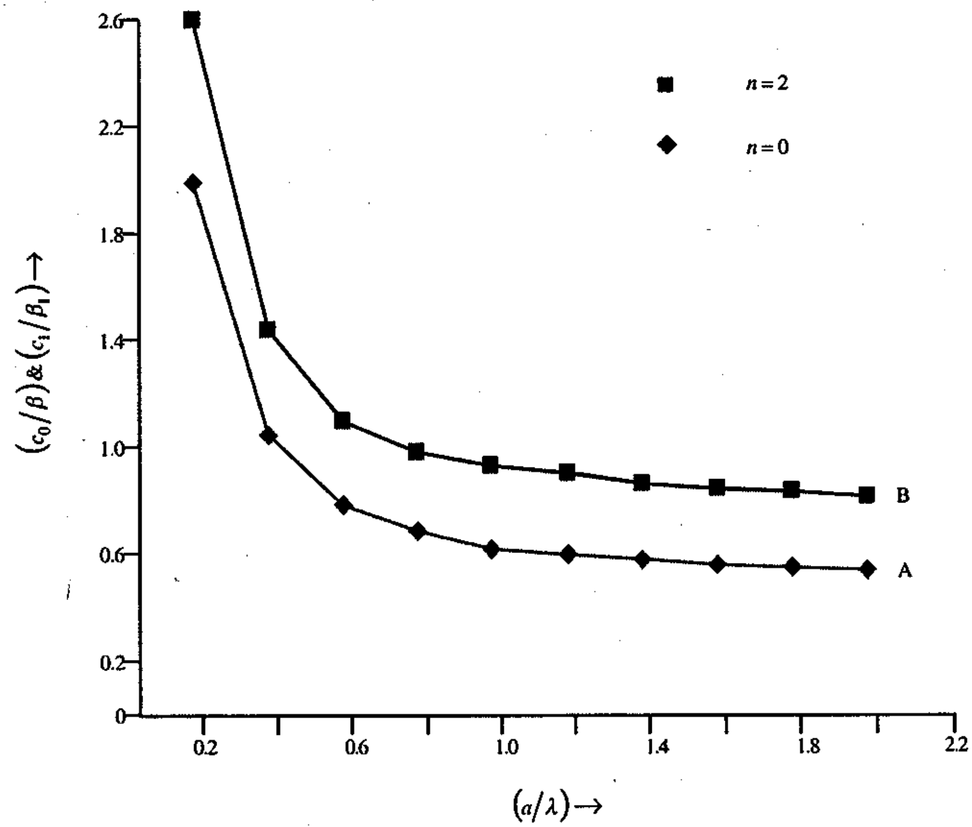

Figure 1. Comparative study of graphs A and B between c_1/β_1 and c_0/β

Table 1. Shows values of c_0/β (*Case A*) and c_1/β_1 (*Case B*) for different values of a/λ

(a/λ)	*Case A* (c_0/β)	*Case B* (c_1/β_1)
0.2	1.9849	2.5680
0.4	1.1662	1.5243
0.6	0.9393	1.2380
0.8	0.8455	1.1206
1.0	0.7985	1.0619
1.2	0.7717	1.0286
1.4	0.7557	1.0080
1.6	0.7441	0.9944
1.8	0.7365	0.9850
2.0	0.7310	0.9782

Eqn (68) is considered and $\xi_2 = 3$, $a = 15$, $\frac{C_{55}}{C_{66}} = 0.9$ are taken and value of (c_1/β_1) for different values of (a/λ) are depicted in Table 1 (*Case B*) and a graph *B* is plotted between these (Fig. 1).

A comparative study of the graphs *A* and *B* clearly shows the effect of non-homogeneity in the absence of magnetic field and homogeneity in the presence of magnetic field.

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