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Free-vibration Characteristics of Laminated Angle-ply Non-circular Cylindrical Shells

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ABSTRACT

This paper deals with the free-vibration behaviour of anisotropic laminated angle-ply non-circular cylindrical shells using finite element approach. The formulation is based on first-order shear deformation theory. The present model accounts for in-plane and rotary inertia effects. A detailed study has been carried out to highlight the effects of shell geometry, cross-sectional properties, lay-up and ply-angles on the natural frequencies of different types of modes of vibration of non-circular elliptical shell structures.

Keywords: Laminated shell, angle-ply, free vibration, non-circular, modes, finite element, elliptical cross section, higher-order shear deformation

NOMENCLATURE

a, b Lengths of semi-major and semi-minor axes of the elliptical cross section

B Strain-displacement matrix

C Total circumferential length

$\dot{d}^e$ Vector of the time derivatives of displacement field variables

E_L, E_T Young's moduli

G_{LT}, G_{TT} Shear moduli

h Total thickness of the shell

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h_k, h_{k+1}	z coordinates of the inner and the outer surfaces of the k^{th} layer
H	Interpolation matrix
k	Layer number
K, M	Global stiffness and mass matrices
K^e, M^e	Element-level stiffness and mass matrices
L	Meridional/axial length of the shell
m	Axial/meridional half-wave number
N	Number of layers
Q_k	Constitutive matrix of k^{th} layer referred to the material principal directions
$\bar{Q}_k$	Constitutive matrix of k^{th} layer referred to the laminated shell axes
R	Principal radius of curvature in the circumferential direction
t	Time
T	Kinetic energy
u, v, w	Displacements along axial, circumferential, and thickness directions, respectively.
u_o, v_o, w_o	Reference surface displacements
U	Strain energy functional
x	Axial/meridional coordinate
y	Circumferential coordinate
z	Thickness/radial coordinate
Z	Matrix relating the velocity components of a generic point with time derivative of d^e
$\bar{Z}$	Matrix relating the middle-surface strains with the strain at any point z
δ	Vector of the DOFs/generalised coordinates
δ^e	Vector of the elemental DOFs/generalised coordinates
δ_i (i = 1, ..., n)	DOFs/generalised coordinates
ε	Total strain
ε_{bm}	Bending and membrane strains
$\varepsilon_i (i = 1, .., 5)$	Middle-surface strain vectors
ε_s	Transverse shear strains
$\varepsilon_{xx}, \varepsilon_{yy}$	Normal strain components
$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	Shear strain components
ν_{LT}	Poisson's ratio
θ_x, θ_y	Rotations of the normal to the middle-surface about the y-axis and x-axis, respectively
ρ_k	Mass density of the k^{th} layer
σ	Stress vector
ω	Frequency
Ω^2, Ω^*	Nondimensional frequency parameters

1. INTRODUCTION

The study of response of shells of revolution under static and dynamic loading situations has received considerable attention in the literature compared to the analysis that deals with shells of non-circular cross section. This is possibly due to the difficulty introduced in the governing equations because of varying nature of the radius of curvature of the non-circular cylinders with the circumferential coordinate. Studies of such shells made of advanced composite materials that are preferred in the design of lightweight and efficient shell structures, are further limited because of the increased complexity due to the inherent directional properties of these materials. Although many structural components of non-circular cross section can be adequately treated as equivalent shells of the revolution, this approach may be unacceptable for shells with significantly non-circular curvature. Thus, there is a growing appreciation of the importance of determining the behaviour, in particular, dynamic characteristics of the laminated composite circular shells with geometrical imperfections and non-circular composite cylindrical shells.

The exhaustive literature[1-5] on dynamic analysis of the circular cylindrical shells/panels has been reviewed. Recently, the limited progress made in understanding the behaviour of the non-circular cylinders has been reviewed by Soldatos[6]. It may be concluded that only a few contributions are available on free-vibration analysis of the anisotropic laminated non-circular cylindrical shells compared to those of isotropic cases, and all these deal with cross-ply cases. Soldatos and Tzivanidis[7], Soldatos[8], and Hui and Du[9] have analysed the free vibrations of multilayered closed oval shells using the classical theory, whereas Noor[10], Soldatos[11], and Kumar and Singh[12] have investigated the same shell structures employing the shear deformation theory. The study on elliptical shells has been done by Suzuki[13], *et al.* based on the classical theory, but Noor[10], and Suzuki[14], *et al.* analysed the same adopting the shear deformation theory. All these investigations have been carried out by the various analytical methods. However, the analysis of angle-ply non-circular shells appears to be scarce in the literature because of increased complexity due to various coupling effects arising from anisotropy. Furthermore, the application of finite element approach, which is a numerical method and easily amenable to solve the problem with complicated geometry and loading situations in analysing the non-circular cylindrical shell structures, has not received adequate attention in the literature.

The purpose of the present study is to analyse the dynamic behaviour of anisotropic laminated angle-ply non-circular cylindrical shells based on first-order shear deformation theory. The analysis is carried out adopting finite element approach. The formulation includes in-plane and rotary inertia effects. The strain-displacement relationship for the shell geometry is accurately accounted for, without assuming the term $z/R<<1$. The accuracy of the present model is validated against the available analytical solutions. A detailed parametric study has been carried out to bring out the effects of eccentricity parameter, thickness, and slenderness ratios, lay-up and ply-angles on the free-vibration characteristics of simply supported non-circular shells with elliptical cross sections.

2. FORMULATION

A laminated composite non-circular cylindrical shell has been considered with the coordinates x along the meridional direction, y along the circumferential direction and z along the thickness direction having

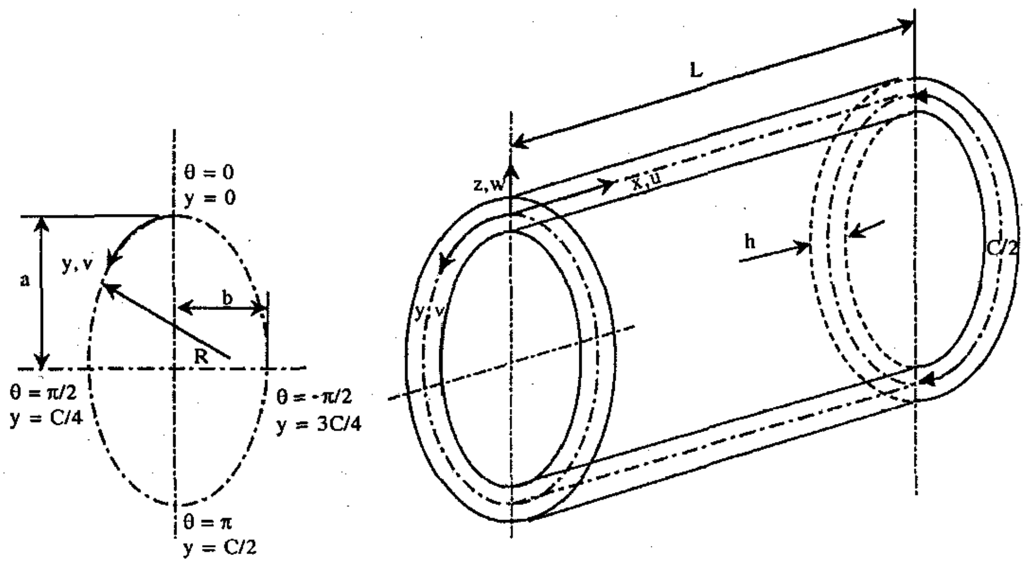

Figure 1. Generalised coordinate system and cross-sectional details of the elliptical shell

origin at the mid-plane of the shell, as shown in the Fig. 1. The displacements u, v, w at a point (x, y, z) are expressed as functions of the mid-plane displacements u_0, v_0, and w_0, and the independent rotations θ_x and θ_y of the normal in the xz and yz planes, respectively as

$$u(x,y,z,t) = u_0(x,y,t) + z\,\theta_x(x,y,t)$$

$$v(x,y,z,t) = v_0(x,y,t) + z\,\theta_y(x,y,t)$$

$$w(x,y,z,t) = w_0(x,y,t) \quad (1)$$

where t is the time.

The strain vector $\{\varepsilon\}$ consisting of bending and membrane strain components $\{\varepsilon_{bm}\} = \{\varepsilon_{xx}\,\varepsilon_{yy}\quad\gamma_{xy}\}^T$, and transverse shear strain terms $\{\varepsilon_s\} = \{\gamma_{xz}\quad\gamma_{yz}\}^T$ is accurately introduced in the formulation and is defined[15] as

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_{bm}\\ \varepsilon_s\end{Bmatrix}=\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}=\begin{Bmatrix}u_{,x}\\ \left(v_{,y}+w/R\right)/\left(1+z/R\right)\\ u_{,y}/\left(1+z/R\right)+v_{,x}\\ u_{,z}+w_{,x}\\ v_{,z}+\left(w_{,y}-v/R\right)/\left(1+z/R\right)\end{Bmatrix} \quad (2)$$

where the subscript comma denotes the partial derivative wrt the spatial coordinate succeeding it.

The principal radius of curvature in the circumferential direction, R is the function of circumferential coordinate y and its variation along the y direction depends on the type of cross section.

Using the kinematics given in Eqn (1), Eqn (2) can be rewritten as

$$\begin{Bmatrix}\varepsilon_{bm}\\ \varepsilon_s\end{Bmatrix}=\left[\bar{Z}\right]\{\varepsilon_1\;\;\varepsilon_2\;\;\varepsilon_3\;\;\varepsilon_4\;\;\varepsilon_5\}^T \quad (3)$$

where

$$\left[\bar{Z}\right]=\begin{bmatrix}1 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{1+z/R} & 0 & 0 & 0 & \frac{z}{1+z/R} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{1+z/R} & 1 & 0 & 0 & \frac{z}{1+z/R} & z & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{1+z/R} & \frac{z}{1+z/R}\end{bmatrix}$$

$$\{\varepsilon_1\}=\begin{Bmatrix}u_{0,x}\\ v_{0,y}+w_0/R\\ u_{0,y}\\ v_{0,x}\end{Bmatrix};\;\{\varepsilon_2\}=\begin{Bmatrix}\theta_{x,x}\\ \theta_{y,y}\\ \theta_{x,y}\\ \theta_{y,x}\end{Bmatrix};\;\{\varepsilon_3\}=\{\theta_x+w_{0,x}\};$$

$$\{\varepsilon_4\}=\begin{Bmatrix}\theta_y\\ w_{0,y}-v_0/R\end{Bmatrix};\;\{\varepsilon_5\}=\{-\theta_y/R\} \quad (4)$$

The constitutive relations for an arbitary layer k, in the laminated shell (x,y,z) coordinate system can be expressed as

$$\{\sigma\}=\{\sigma_{xx}\;\;\sigma_{yy}\;\;\tau_{xy}\;\;\tau_{xz}\;\;\tau_{yz}\}^T$$
$$=\left[\bar{Q}_k\right]\{\varepsilon_{xx}\;\;\varepsilon_{yy}\;\;\gamma_{xy}\;\;\gamma_{xz}\;\;\gamma_{yz}\}^T \quad (5)$$

where the terms of $\left[\bar{Q}_k\right]$ matrix of k^{th} ply are referred to the laminated shell axes and can be obtained from the $[Q_k]$ corresponding to the fibre directions with the appropriate transformation, as outlined in the literature[16]. $\{\sigma\}$ and $\{\varepsilon\}$ are the stress and the strain vectors, respectively. The superscript T refers to the transpose of a matrix/vector.

The kinetic energy of the shell is given by

$$T(\delta)=\frac{1}{2}\iint\left[\sum_{k=1}^{N}\int_{h_k}^{h_{k+1}}\rho_k\{\dot{u}\ \dot{v}\ \dot{w}\}\{\dot{u}\ \dot{v}\ \dot{w}\}^T\left(1+\frac{z}{R}\right)dz\right]dx\,dy \tag{6}$$

where ρ_k is the mass density of the k^{th} layer, h_k, h_{k+1} are the z coordinates of the laminate corresponding to the bottom and top surfaces of the k^{th} layer, $\{\delta\}=\{\delta_1, \delta_2, \ldots, \delta_i, \ldots, \delta_n\}^T$ is the vector of the DOFs/generalised coordinates, and N is the number of layers. A dot over the variables represents the partial derivative wrt time.

Using the kinematics given in Eqn (1), Eqn (6) can be rewritten as

$$T(\delta)=\frac{1}{2}\iint\left[\sum_{k=1}^{N}\int_{h_k}^{h_{k+1}}\rho_k\{\dot{d}^e\}^T\{Z\}^T\{Z\}\{\dot{d}^e\}\left(1+\frac{z}{R}\right)dz\right]dx\,dy \tag{7}$$

where

$$\{\dot{d}^e\}^T=\{\dot{u}_0\ \dot{v}_0\ \dot{w}_0\ \dot{\theta}_x\ \dot{\theta}_y\}$$

and

$$[Z]=\begin{bmatrix}1&0&0&z&0\\0&1&0&0&z\\0&0&1&0&0\end{bmatrix}$$

The strain energy functional U is given by

$$U(\delta)=\frac{1}{2}\iint\left[\sum_{k=1}^{N}\int_{h_k}^{h_{k+1}}\{\sigma\}^T\{\varepsilon\}\left(1+\frac{z}{R}\right)dz\right]dx\,dy \tag{8}$$

For the sake of convenience, the kinetic and the strain energy functionals at the elemental level may be rewritten as

$$T(\delta^e)=\frac{1}{2}\{\dot{\delta}^e\}^T\left[M^e\right]\{\dot{\delta}^e\} \tag{9}$$

$$U(\delta^e)=\frac{1}{2}\{\delta^e\}^T\left[K^e\right]\{\delta^e\} \tag{10}$$

The elemental mass $\left[M^e\right]$ and stiffness $\left[K^e\right]$ matrices involved in the Eqns (9) and (10) can be defined as

$$\left[M^e\right]=\iint\left[\sum_{k=1}^{N}\int_{h_k}^{h_{k+1}}\rho_k\{H\}^T\left[Z\right]^T\left[Z\right]\{H\}\left(1+\frac{z}{R}\right)dz\right]dx\,dy \tag{11a}$$

$$\left[K^e\right]=\iint\left[\sum_{k=1}^{N}\int_{h_k}^{h_{k+1}}\left[B\right]^T\left[\bar{Z}\right]^T\left[\bar{Q}_k\right]^T\left[\bar{Z}\right]\left[B\right]\left(1+\frac{z}{R}\right)dz\right]dx\,dy \tag{11b}$$

and $\{\delta^e\}$ is the vector of the elemental DOFs/ generalised coordinates.

The elemental-level governing equations, obtained using Lagrangian equations of motion, are:

$$\left[M^e\right]\{\ddot{\delta}^e\}+\left[K^e\right]\{\delta^e\}=\{0\} \tag{12}$$

The coefficients of mass and stiffness matrices involved in governing Eqn (12) can be rewritten as the product of the term having thickness coordinate z alone and the terms containing x and y. In the present study, while performing the integration, the terms having thickness coordinate z are explicitly integrated, whereas the terms containing x and y are evaluated using full integration with 5 x 5 points Gauss integration rule.

Following the standard finite element assembly procedure[17], the governing equations for the free vibration of the shell are obtained as

$$\left[M\right]\{\ddot{\delta}\}+\left[K\right]\{\delta\}=\{0\} \tag{13}$$

where $[M]$ and $[K]$ are the global mass and stiffness matrices.

The solutions of Eqn (13) can be obtained using standard eigenvalue extraction procedures.

3. ELEMENTS DESCRIPTION

In the present study, a C^0 continous, eight-noded serendipity quadrilateral shear flexible shell element with five nodal DOFs (u_0, v_0, w_0, θ_x, θ_y) developed, based on field consistency approach[18], has been employed. If the interpolation functions for an eight-noded element are used directly to interpolate the five field variables (u_0, v_0, w_0, θ_x, θ_y) in deriving the membrane and shear strains, the element will lock and show oscillations in the membrane and the shear stresses. Field consistency requires that the membrane and the transverse shear strains must be interpolated in a consistent manner. Thus, the w_o term in the expression for membrane strain $\{\varepsilon_1\}$ given in Eqn (4) has to be consistent with the field functions $v_{o,y}$, as suggested by Pratap[18]. Similarly, the terms θ_x and (v_0, θ_y) in the expression for transverse shear strains $\{\varepsilon_3\}$ and $\{\varepsilon_4\}$ defined in Eqn (4) be consistent with the field functions $w_{o,x}$ and $w_{o,y}$, respectively. This is achieved using a field-redistributed substitute shape function to interpolate those specific terms that must be consistent. The element thus derived is tested for its basic properties and is found free from the rank deficiency, shear/membrane locking, and poor convergence syndrome[19].

4. RESULTS & DISCUSSION

In this study, layered elliptical cylinders are investigated as examples of non-circular cylinders with doubly symmetric cross sections. The radius of curvature of the middle surface for elliptical cross section is described[14] as

$$R = (b^2 / R_o)(1+\mu_o \cos 2\theta)^{-3/2}$$

where

$$R_o = [(a^2 + b^2)/2]^{1/2}$$

is the representative radius, $\mu_o = (a^2-b^2)/(a^2+b^2)$; and θ is a variable that denotes an angle between the tangent at the origin of y (circumferential coordinate) and the one at any point on the centreline (Fig. 1). The parameters a and b are the semi-major and the semi-minor axes of the elliptical cross section, respectively.

Since the study is concerned with angle-ply case, the complete shell model is considered for the analysis. To improve the estimation of the transverse shear strain energy, the value for the shear correction factor is taken as 5/6. Furthermore, as the element is based on field-consistent approach, all the energy contributions are evaluated using exact numerical integration scheme. Based on progressive mesh refinement, a 64 x 12 mesh (circumferential and meridional directions) is found to be adequate to model the complete shell for the present analysis. The formulation developed herein is validated for cross-ply non-circular cylindrical shells for which analytical solutions are available in the literature[12,14].

These results are shown in Tables 1 and 2 and are in good agreement. Furthermore, all the numerical results obtained are associated with $m = 1$ axial half-wave number. The effects of various parameters, such as thickness and length ratios (R_o/h and L/R_o), eccentricity parameter (a/b), lay-up and ply-angles on the natural frequencies ($\Omega^2 = \rho\omega^2 R_o^2/E_o$; $E_o = E_L/[12(1-\nu_{LT}\nu_{TL})]$) corresponding to different types of spatially fixed asymmetric vibration modes are examined. Since the conventional way of identifying the circumferential modes, based on number of crossings is not valid due to the variable radius of curvature causing many modes that have the same number of crossings around the circumference but corresponds to different frequencies[20], first 12 frequencies in the order from below and their associated asymmetric modes are therefore highlighted for the present study. The various modes are symmetry-symmetry (SS), symmetry-antisymmetry (SA), antisymmetry-symmetry (AS), and antisymmetry-antisymmetry (AA), classified based on whether the modes are symmetric or antisymmetric wrt the axes passing through the points $y = 0$ and C/2, and y = C/4 and 3C/4, respectively (Fig. 1) where C is the total circumferential length of the shell. The material properties used, unless otherwise specified, are:

Table 1. Comparison of frequency parameter Ω^2 for three-layered cross-ply (90°/0°/90°) elliptical shell

a/b	K		SS modes*		AA modes*		SA modes			AS modes		
			1st	2nd	1st	2nd	1st	2nd	3rd	1st	2nd	3rd
1.53	0.5	Present	0.1658	3.0450	0.2265	3.0557	0.0624	1.0251	6.3913	0.0333	1.0230	6.3814
		Ref.[13]	0.1670	3.0540	0.2270	3.0660	0.0630	1.0290	-	0.0340	1.0280	-
	3.0	Present	1.1216	3.8628	1.2612	3.8996	1.7681	3.4298	7.2371	1.4162	2.0683	7.2364
		Ref.[13]	1.1440	3.9090	1.2850	-	1.8000	3.4460	7.3080	1.4390	2.0980	-
1.22	0.5	Present	0.1691	3.0682	0.1911	3.0689	0.0560	1.0455	6.3878	0.0415	1.0453	6.3876
		Ref.[13]	0.1710	3.0790	0.1920	3.0800	0.0560	1.0500	6.4050	0.0420	1.0500	6.4050
	3.0	Present	1.3082	3.8959	1.3517	3.8991	1.8870	2.8981	7.2154	1.7885	2.1386	7.2154
		Ref.[13]	1.3310	3.9440	1.375	3.9470	1.921	2.916	7.2750	1.8160	2.1640	7.2750

*S = Symmetric, A = Antisymmetric, $K = m\pi R_o/L$, $R_o/h = 6$, $E_L = 138$ GPa, $E_T = 8.9$ GPa, $G_{LT} = 7.1$ GPa, $G_{TT} = 3.45$ GPa, $\nu_{LT} = 0.30$

Table 2. Comparison of the lowest frequency parameter $\Omega^* (= \sqrt{\rho\omega^2 R_0^2 / E_L}$) for cross-ply $(0°/90°)_{N/2}$ oval shells

No. of layers N	$a/b = 1.14$				$a/b = 1.75$			
	SS & SA modes		AS & AA modes		SS & SA modes		AS & AA modes	
	Ref. [12]	Present	Ref. [12]	Present	Ref. [12]	Present	Ref. [12]	Present
2	0.1099	0.1085	0.1099	0.1085	0.0646	0.0640	0.0646	0.0640
4	0.1223	0.1210	0.1223	0.1234	0.0725	0.0719	0.0725	0.0719
6	0.1243	0.1229	0.1244	0.1256	0.0738	0.0731	0.0738	0.0731
8	0.1250	0.1236	0.1251	0.1263	0.0741	0.0735	0.0741	0.0735
10	0.1254	0.1239	0.1254	0.1267	0.0744	0.0737	0.0743	0.0737

$L/R_o = 1.0$, $R_o/h = 100$, $E_T/E_L = 0.1$, $G_{LT}/E_L = G_{TT}/E_L = 0.05$, $\nu_{LT} = 0.25$, R_o: radius of equivalent circular shell

$E_L/E_T = 40$, $G_{LT}/E_T = 0.6$, $G_{TT}/E_T = 0.5$, $\nu_{LT} = 0.25$, $E_T = 1$ GPa, $\rho = 1500$ kg/m³

where E, G, ν and ρ are the Young's modulus, shear modulus, Poisson's ratio, and density, respectively. The subscripts L and T refer to the longitudinal and the transverse directions, respectively, wrt the fibres. All the layers are of equal thickness and the ply-angle (β) is measured wrt the x-axis (meridional axis).

The simply supported boundary conditions considered here are:

$$u_0 = v_0 = w_0 = \theta_y = 0 \qquad \text{at } x = 0,\ L$$

Next, by varying the ply-angle of two-and eight-layered [$\beta^0/-\beta^0$ and $(\beta^0/-\beta^0)_4$] laminated thin elliptical shells (L/R_o = 0.5 and 1.0; R_o/h = 100), the free-vibration characteristics are investigated assuming two values of eccentricity

Table 3. Frequency parameters Ω^2 of two-layered and eight-layered thin angle-ply elliptical shells with a/b = 1.5

Mode		Two-layered (β/-β) shell					Eight-layered $(\beta/-\beta)_4$ shell				
		β = 15°	30°	45°	60°	75°	β = 15°	30°	45°	60°	75°
SS	1st	0.1961	0.2866	0.5248	0.4768	0.3391	0.2571	0.4202	0.8617	0.8022	0.4804
	2nd	0.2175	0.3432	0.7026*	0.6521	0.5191	0.3052	0.5548	0.9988	1.1198	0.7383
	3rd	0.2435	0.4113	0.8375*	0.8382	0.7055	0.3702	0.7296	1.2327	1.4372	1.1198
AA	1st	0.1968	0.2884	0.5374*	0.4768	0.3391	0.2787	0.4821	0.8828	0.8023	0.4804
	2nd	0.2245	0.3586	0.6448*	0.6521	0.5191	0.3357	0.6373	1.0995	1.1220	0.7387
	3rd	0.2609	0.4493	0.7781*	0.8384	0.7058	0.4085	0.8317	1.3855	1.4678	1.1218
SA	1st	0.1968	0.2884	0.5373*	0.4771	0.3403	0.2787	0.4821	0.8828	0.8038	0.4778
	2nd	0.2245	0.3586	0.6450*	0.6472	0.5072	0.3357	0.6373	1.0998	1.1114	0.7622
	3rd	0.2609	0.4493	0.7784*	0.8699	0.7496	0.4085	0.8317	1.3840	1.4954	1.0530
AS	1st	0.1961	0.2866	0.5248	0.4771	0.3403	0.2571	0.4202	0.8616	0.8037	0.4778
	2nd	0.2175	0.3432	0.7024*	0.6472	0.5071	0.3052	0.5548	0.9987	1.1092	0.7618
	3rd	0.2435	0.4113	0.8346*	0.8695	0.7494	0.3702	0.7296	1.2335	1.4721	1.0497

*Modes with slight deviation from perfect symmetry/antisymmetry, L/R_o = 0.5, R_o/h = 100

Table 4. Frequency parameters Ω^2 of two-layered (β/-β) thin angle-ply elliptical shells with a/b = 2.5

Mode		L/R_o = 0.5					L/R_o = 1.0				
		β = 15°	30°	45°	60°	75°	β = 15°	30°	45°	60°	75°
SS	1st	0.1400	0.1476	0.2263	0.2751	0.1895	0.0258	0.0560	0.0804	0.0692	0.0586
	2nd	0.1548	0.1798	0.2972*	0.3856	0.3206	0.0360	0.0767*	0.1248	0.1275	0.1246*
	3rd	0.1813	0.2415	0.5715*	0.5081	0.5152	0.0502	0.1062*	0.1875*	0.2269	0.2163
AA	1st	0.1452	0.1562	0.2344*	0.2753	0.1896	0.0260	0.0564*	0.0804	0.0692	0.0587
	2nd	0.1665	0.2059	0.4849*	0.3860	0.3223	0.0388	0.0908*	0.1269*	0.1277	0.1255
	3rd	0.1994	0.2845	0.6688*	0.5615	0.5302	0.0590	0.1268*	0.1896	0.2306	0.2196
SA	1st	0.1452	0.1562	0.2344*	0.2754	0.1892	0.0260	0.0564*	0.0805	0.0695	0.0584
	2nd	0.1665	0.2059	0.4852*	0.3858	0.3249	0.0388	0.0908*	0.1264*	0.1293	0.1219
	3rd	0.1994	0.2845	0.6709*	0.5610	0.5245	0.0590	0.1269*	0.1927	0.2103	0.2547
AS	1st	0.1400	0.1476	0.2263	0.2751	0.1891	0.0258	0.0560	0.0805	0.0696	0.0583
	2nd	0.1548	0.1798	0.2971*	0.3853	0.3232	0.0360	0.0766*	0.1241	0.1291	0.1215*
	3rd	0.1813	0.2415	0.5710*	0.5098	0.5081	0.0502	0.1063*	0.1896*	0.2091	0.2475

*Modes with slight deviation from perfect symmetry/antisymmetry, L/R_o = 0.5 and 1.0, R_o/h = 100

Table 5. Frequency parameters Ω^2 of two-layered (β/-β) moderately thick angle-ply elliptical shells with a/b = 1.5

Mode		$L/R_o = 0.5$					$L/R_o = 1.0$				
		β = 15°	30°	45°	60°	75°	β = 15°	30°	45°	60°	75°
SS	1st	3.9149	3.4250	2.9913	2.7957	2.3709	0.5884	0.6338	0.9037	1.0811	0.8230
	2nd	4.2204	4.0775	3.9826	4.1459	3.4945	0.7334	0.9604	1.4906	1.7083	1.2165
	3rd	4.6354	5.0228	5.6827	5.9864	4.6041	1.0673	1.6183*	2.2540*	2.4031	2.6903
AA	1st	4.0464	3.7111	3.3896	3.2304	2.4670	0.6440	0.7453	1.0399*	1.1077	0.8497
	2nd	4.3927	4.4521	4.6616	5.0140	4.1675	0.8147	1.1661	1.9169*	2.2440*	1.3769
	3rd	4.9068	5.4197	6.1894*	6.8228	5.2827	1.1217	1.7520	2.5212*	2.6670	-
SA	1st	4.0481	3.7127	3.3831	3.2331	2.5064	0.6401	0.7421	1.0431*	1.1643*	0.7116
	2nd	4.4305	4.5217	4.7309	4.9817	3.6749	0.8691	1.2344	1.8831*	1.7828*	1.6587
	3rd	4.7752	5.4252	6.7821*	6.9518*	5.8993	1.1641	2.4577	2.9425	3.4905	4.0784
AS	1st	3.9148	3.4251	2.9919	2.7952	2.3835	0.5888	0.6343	0.8973	1.1129	0.7001
	2nd	4.2080	4.0604	3.9902	4.1418	3.4068	0.7320	0.9571	1.4842	1.6270	1.6379
	3rd	4.6129	4.8875	5.3827	6.0914	5.4065	0.9263	1.4216	2.3089	-	4.0845

*Modes with slight deviation from perfect symmetry/antisymmetry, L/R_o = 0.5 and 1.0, R_o/h = 10

Table 6. Frequency parameters Ω^2 of two-layered (β/-β) moderately thick angle-ply elliptical shells with a/b = 2.5

Mode		$L/R_o = 0.5$					$L/R_o = 1.0$				
		β = 15°	30°	45°	60°	75°	β = 15°	30°	45°	60°	75°
SS	1st	3.8232	3.1912	2.3935	1.6893	1.3514	0.5132	0.4482	0.4743	0.5763	0.4518
	2nd	4.1307	3.8372	3.4021	3.0707	2.3636	0.6726	0.7841	1.0366	1.0471	0.9363
	3rd	4.7333	5.0051	5.3219	5.3020	4.3707	1.0678	1.5346*	4.0328	2.3682	6.0361
AA	1st	3.9449	3.4538	2.7962	2.2211	1.7098	0.5739	0.5708	0.6672*	0.7416	0.5851
	2nd	4.3586	4.3110	4.1875	4.1616	3.4103	0.7962	1.0726	1.6498*	1.7826	2.7943
	3rd	5.0209	5.6354	6.4631	8.7656*	5.3153	1.1957	3.1727*	4.2527	2.7240	6.0935
SA	1st	3.9462	3.4560	2.7981	2.2255	1.6729	0.5724	0.5708	0.6633	0.6963	0.4890
	2nd	4.3881	4.3477	4.2354	4.0880	3.1706	0.8295	1.0963	2.9126*	1.5353	1.6122
	3rd	5.1812	5.8263	6.6346	6.6312	5.9362*	1.4050	3.8062	-	2.2688	4.1537
AS	1st	3.8231	3.1910	2.3935	1.6874	1.3435	0.5137	0.4484	0.4704	0.5605	0.4395
	2nd	4.1227	3.8262	3.3879	3.0807	2.5203	0.6690	0.7797	1.0788*	1.2529	1.1549
	3rd	4.6550	4.9119	5.2184	5.4820	4.4811	0.9637	1.4614	3.0220	3.7243	4.2319

*Modes with slight deviation from perfect symmetry/antisymmetry, L/R_o = 0.5 & 1.0, R_o/h = 10 parameter (a/b = 1.5, 2.5). The results obtained for the first few natural frequencies of various types of asymmetric modes of vibration are presented in the Tables 3 and 4. It is observed from the Table 3 that the 15° ply-angle case results in the lowest frequency values of all asymmetric modes of vibration and the next higher one is mostly yielded by 30° case. The maximum values occur either for the ply-angle 45° or for 60° case, depending on the frequency order and the type of asymmetric mode of vibration. The behaviour of eight-layered case is qualitatively similar to that of two-layered one but the frequency value increases due to the weakening of bending-stretching coupling introduced in the laminate. It can be further noticed from Table 4 that, with increase in L/R_0 value, the frequency value, in general, significantly decreases and the ply-angle at which maximum frequency was obtained shifted to 75°, especially for the higher modes. Furthermore, it can be opined from Tables 3 and 4 that, the increase in eccentricity value affects the frequencies quantitatively but the nature of variation of the frequency parameter does not change. In general, it can be viewed that, for higher ply-angle cases, the frequency order can change with the type of vibration modes, irrespective of the number of layers and the eccentricity parameter.

For a moderately thick case (R_0/h =10), the frequency parameter values of different asymmetric modes of vibration are highlighted in Tables 5 and 6 for the two values of eccentricity (a/b = 1.5 and 2.5) and length (L/R_0 = 0.5 and 1.0) parameters. It is evident from these tables that, for the higher length ratio (L/R_0 = 1.0) considered here, the variation of frequency parameter is, in general, qualitatively similar to those of thin shell cases whereas no such conclusion can be drawn for thick shell cases (L/R_0 = 0.5), wherein the behaviour depends on mode number/type of asymmetric mode of vibration. However, for most of the frequencies highlighted here, it appears that 75° ply-angle cases can predict the lowest values and 15° ply-angle cases yield the maximum values, irrespective of eccentricity ratio. It is also observed that, with increase in eccentricity parameter value, the first two frequencies of each type of asymmetric mode, decrease, and the rate of decrease is high for the smaller length ratio case compared to the bigger length ratio case.

Finally, the circumferential variations of the first two mode shapes of various types of asymmetric vibrations are shown in the Figs 2 and 3 for thin and moderately thick shell cases. It is observed from the mode shape analysis of the cross-ply case[14] that cos ($n\theta$) terms alone contribute for SS or SA cases, depending on n even or odd, while sin ($n\theta$) terms alone participate for AA or AS cases, depending on n even or odd. However, it has been found (not shown here) that the contributions of both cos ($n\theta$) and sin ($n\theta$) (n even and odd) are noticeable in a mode shape of any type of asymmetric vibration of angle-ply case. It is seen from Fig. 2 that the ply-angle can affect the peak values along the circumferential direction and also their spatial occurrence. It can also be viewed from the Fig. 2 that some mode shapes show deviation from perfect symmetry/antisymmetry for certain ply-angles, for instance the second AA mode of 30° cases about line joining y/C = 0.25 and 0.75. However, for thick shell situation, (Fig. 3) although there are some changes in the peak amplitude values with increase in ply-angle, their spatial occurrence does not change noticeably compared to those of thin shell cases. It is also seen that the peak amplitudes in the inward and the outward directions are different because of variable curvature of the non-circular shell, leading to the coupling of all the vibration modes of a corresponding circular cylindrical shell, and this coupling can be strong with the increase in the non-circularity of the middle surface cross section.

5. CONCLUSIONS

The free-vibration characteristics of laminated angle-ply non-circular, elliptical shells have been analysed based on the first-order shear deformation theory through finite element approach. The effect of eccentricity, ply-angles, and lay-up on the values of natural frequency parameter has been demonstrated. From the detailed parametric study, the following observations have been be made:

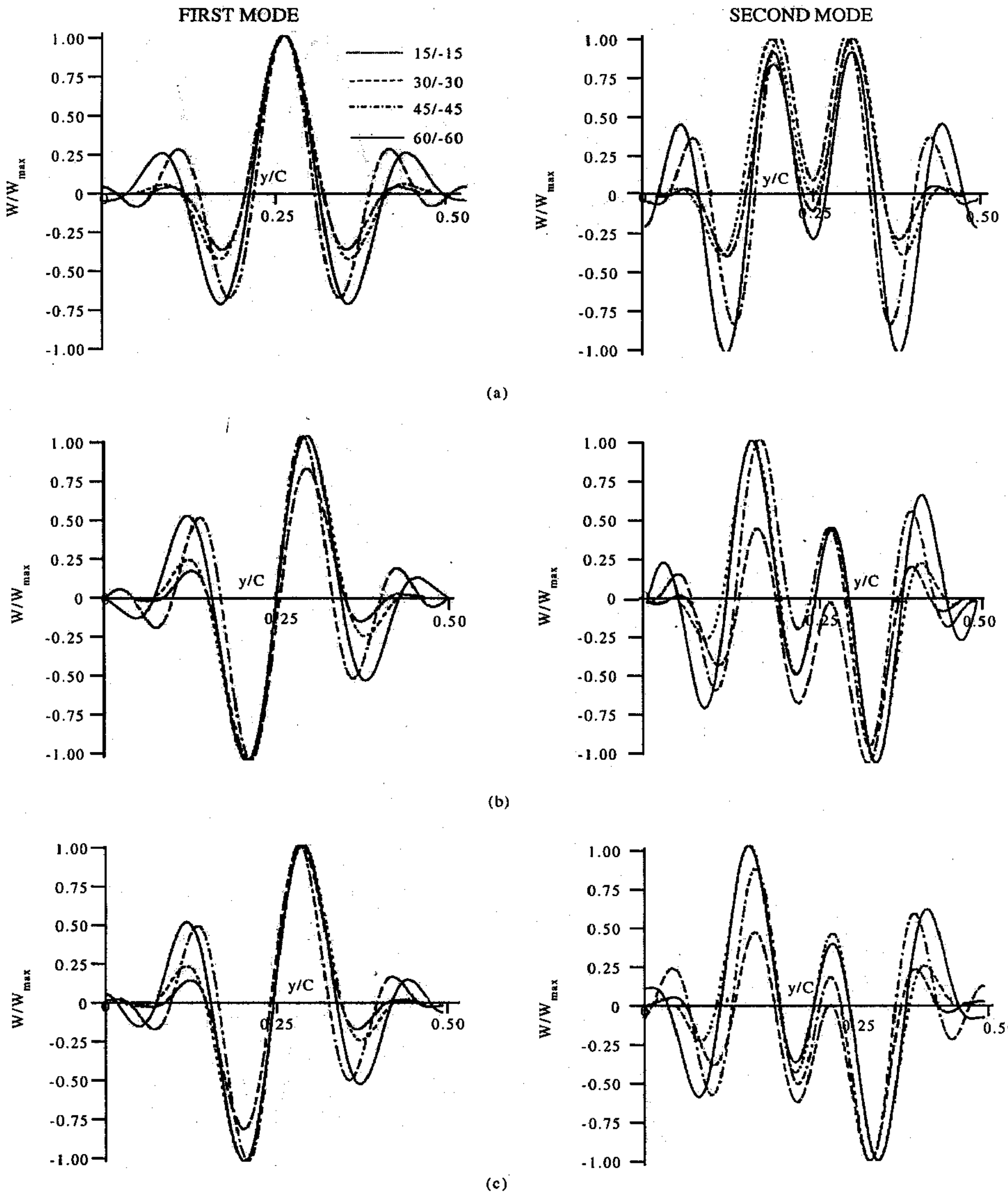

Figure 2. Mode shapes of two-layered (β/–β) thin elliptical shell ($R_e/h = 100$, $L/R_e = 1.0$, $a/b = 2.5$): (a) first and second SS modes, (b) first and second AA modes, and (c) first and second SA modes.

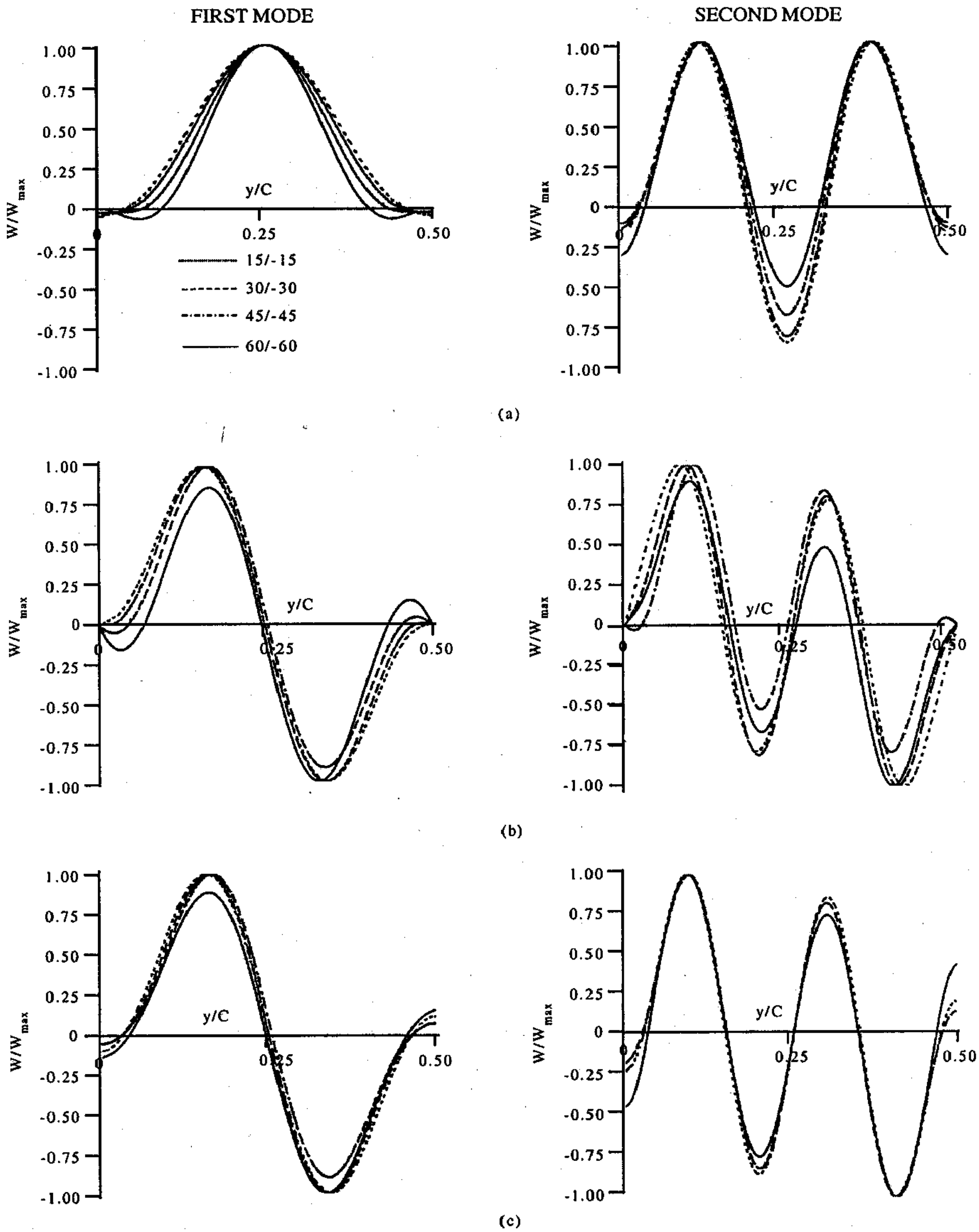

Figure 3. Mode shapes of two-layered (β/–β) moderately thick elliptical shell (R_e/h = 10, L/R_e = 1.0, a/b = 2.5): (a) first and second SS modes, (b) first and second AA modes, and (c) first and second SA modes.

(a) For a thin shell, in general, the lowest frequency values corresponding to different asymmetric modes of vibration are predicted for ply-angle of 15°, whereas the maximum values occur for 45° or 60° ply-angle cases. But, it strongly depends on the value of the length parameter for thick shells.

(b) The increase in the number of layers and non-circularity does not affect the qualitative variation of the frequency parameter.

(c) For thick shell cases, ply-angle of 75° yields minimum frequency values and ply-angle of 15° case yields maximum values, for most of the modes of vibrations considered here.

(d) With increase in eccentricity parameter value, the frequency value decreases and the rate of decrease is more for bigger length ratio case compared to the case of smaller length ratio case.

(e) The ply-angle can affect the peak values of the inward and the outward displacements and their spatial occurrence in the shell's circumferential direction, depending on the values of thickness parameter. To some extent, it can also disturb the symmetric/antisymmetric nature of the mode shape.

(f) Unlike cross-ply case, the contributions of both cos ($n\theta$) and sin ($n\theta$) (n even and odd) are noticeable in a mode shape of any type of asymmetric vibrations of angle-ply cases.

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