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SHORT COMMUNICATION

Response of a Cracked Cantilever Beam to Free and Forced Vibrations

V. M. Radhakrishnan

Indian Institute of Technology Madras, Chennai–600 036

ABSTRACT

Cracks present in machine parts affect their vibrational behaviour like the fundamental frequency and the resonance. In this paper, the resonance response of a cracked cantilever rectangular beam has been studied based on fracture mechanics quantities like strain energy release rate, stress intensity factor and compliance. The spring stiffness and the fundamental frequency decrease with increase in crack length. The amplitude of vibration increases and the occurrence of resonance gets shifted with increase in crack length.

Keywords: Vibration, resonance, beam, crack, cantilever beam

NOTATIONS

a	Crack length
A	$(1 - a/W)$
C	Compliance
E	Young's modulus
G	Crack extension force
I	Moment of inertia
In	Integral
k	Spring stiffness
K	Stress intensity factor
L	Length of the beam
M	Mass of the beam
Mb	Bending moment
P	Force
t	Thickness of the beam
W	Width of the beam
Y	Deflection/amplitude
Yo	Static deflection without crack
$Yo\ (a)$	Deflection/ amplitude with crack
α	$(A^{-3} - A^3)$
β	L/W
ωo	Fundamental frequency
$\omega\ (a)$	Fundamental frequency with crack

1. INTRODUCTION

Fatigue-type of loading of engineering structures and machine parts is likely to introduce cracks at the highly stressed regions. Many times, manufacturing methods like welding may also introduce crack-like defects. In engineering machinery and civil structures, members in the

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form of cantilever-type of beams are widely used. Common examples are propeller shaft, turbine blades, cantilever bridges, tall building structures, etc. Cracks are likely to nucleate and grow in the tensile stress regions of the beam, as indicated in Fig. 1. The main consequence of these cracks is to alter the

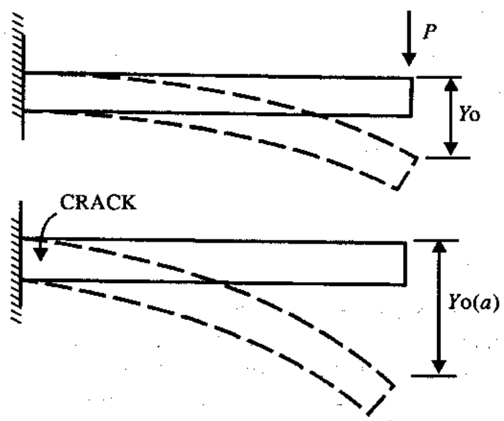

Figure 1. Cantilever beam without a crack and with a crack at the fixed end.

stiffness of the beam, leading to change in the fundamental frequency. In the absence of a crack, the fundamental frequency for a cantilever beam is given by

$$\omega_o = \frac{3.53}{\sqrt{3}}\sqrt{\frac{3EI}{ML^3}} = 2.04\sqrt{\frac{k}{M}} \qquad (1)$$

The spring stiffness decreases with an increase in crack length, a. As a result of this, the fundamental frequency also decreases. Thus, the vibration response of the cantilever beam gets altered due to the formation and growth of the crack.

Many investigators[1-5] have studied the effect of defects and cracks in different vibrating structures. A finite element technique has been widely used to analyse the stiffness and to develop models to estimate the frequencies. Parthi and Behera[6] have investigated the wave forms of different modes of a cracked shaft, using stress intensity factors.

In the present study, the effect of a crack at the fixed end and at the mid-span on the free and forced vibration response of a cantilever beam has been analysed using the fracture mechanics approach. Since cantilever-type of structures are common in many engineering components, this method will be useful to the design engineers, who base their design on the fracture mechanics concepts. The assumptions made in the derivation of fracture mechanics quantities are also applicable in the present analysis.

2. STRAIN ENERGY RELEASE RATE & STRESS INTENSITY FACTOR

Consider an elastic body containing a crack of length, a, as shown in Fig. 2. The load and the corresponding loadpoint deflection are P and Y, respectively. Now, if the crack extends by a small distance, da, the load required to cause the same deflection gets reduced and the difference in the elastic strain energy, dU, goes to make the crack propagate through the distance, da. Thus, the elastic strain energy release rate[7], dU/da, is given by

$$\frac{dU}{da} = \frac{p^2}{2}\frac{dC}{da} \qquad (2)$$

where C is the compliance, which is the inverse of the spring stiffness of the material in the presence of a crack as indicated in Fig. 2.

The crack extension force, G, is the strain energy release rate per unit thickness of the material. It is also related to the stress intensity factor, K_I, through the Young's modulus of the material. Thus, one has:

$$G = \frac{1}{t}\frac{dU}{da} = \frac{K_I^2}{E} \qquad (3)$$

The above equation leads to

$$\frac{dC}{da} = \frac{2t}{E}\frac{K_I^2}{P^2} \qquad (4)$$

Thus, the change in the compliance is related to the stress intensity factor.

The stress intensity factor of a cracked body depends on the shape and size of the body, the type of loading and the length of the crack, and

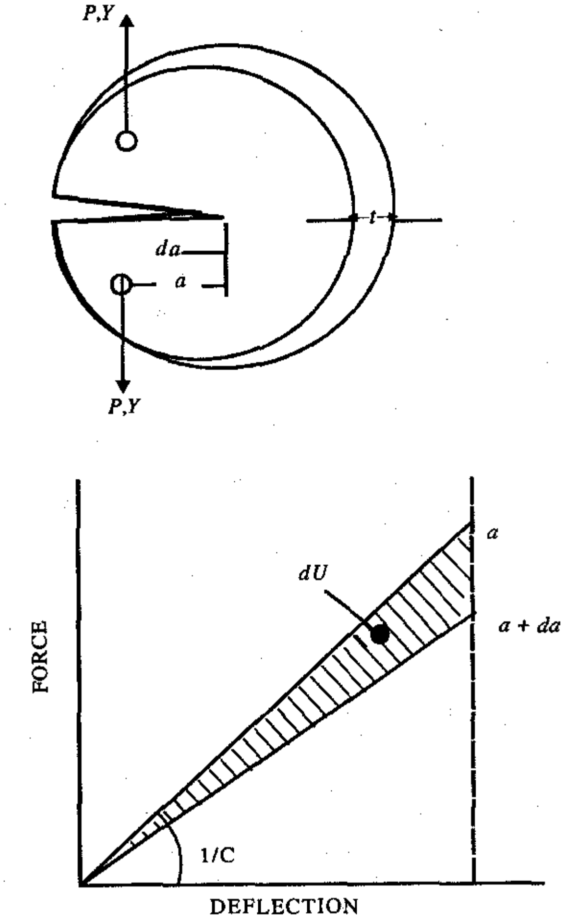

Figure 2. Strain energy release and compliance

its ratio wrt the width (a/W). Thus, for a cantilever beam with a crack at the top surface[8], the stress intensity factor is given by

$$K_I = \frac{4.12Mb}{t\,W^{3/2}}[\alpha]^{1/2} \tag{5}$$

3. COMPLIANCE & SPRING STIFFNESS

With the bending moment (Mb) given as $Mb = PL$, one gets:

$$\int_{Co}^{C} dC = \frac{2(4.12)^2(L)^2}{E\,t\,W^2}[\mathrm{In}] \tag{6}$$

where

$$\mathrm{In} = \int_0^{a/W}[\alpha]\,da/W \tag{7}$$

Expanding the term α and integrating, the values 'In' can be obtained for different crack lengths. The relationship between the integral (In) and the crack length, a/W, is shown in Fig. 3.

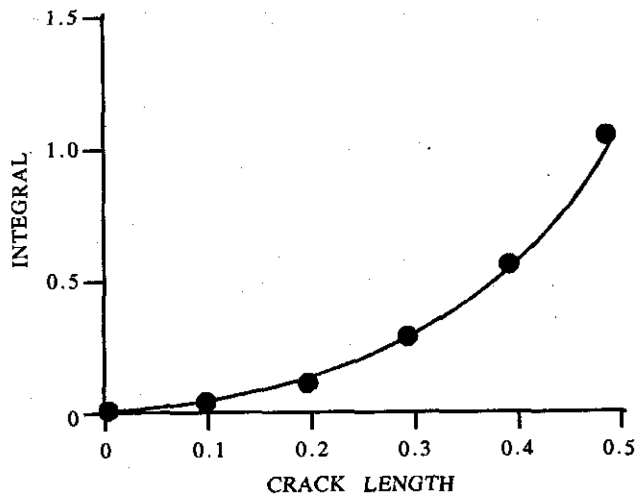

Figure 3. Variation of integral with crack length

Thus, one gets the compliance as

$$C\text{-}Co = \frac{34\,L^2}{E\,t\,W^2}[\mathrm{In}] \tag{8}$$

The compliance C is in m/N. Co is the value ($1/ko$) at the free end of the cantilever beam when $a/W = 0$, i.e.

$$Co = \frac{L^3}{3EI} \tag{9}$$

Thus the compliance C is given by

$$C = \frac{34\,L^2}{E\,t\,W^2}[\mathrm{In}] + \frac{L^3}{3EI} \tag{10}$$

or, with the moment of inertia, $I = t\,W^3/12$, one gets:

$$C = \frac{34L^2}{E\,t\,W^2}[\mathrm{In}] + \frac{4\,L^3}{3E\,t\,W^3} \tag{11}$$

The compliance can be given in the non-dimensional form as

$$E\,t\,C = \beta^2\,(34\ \mathrm{In} + 4\,\beta) \tag{12}$$

where $\beta = L/W$ and for a beam without a crack

$$E\ t\ Co = \beta^2 (4\ \beta)$$

And so, one has the ratio compliance Ca in the presence of the crack to the compliance Co without a crack as

$$\frac{Ca}{Co} = \frac{34\text{In} + 4\beta}{4\beta} \quad (13)$$

Table 1. Values of integral (In), compliance (C), the spring stiffness (k), the fundamental frequency [$\omega(a)$] and the static deflection [$Yo(a)$] for $\beta = 8$

a/W	In	$E\,t\,C$	$k(a)/E\,t$ 10^{-4}	$\omega(a)/\omega o$	$Yo(a)/Yo$
0.0	0.000	2048	4.88	1.00	1.00
0.1	0.031	2115	4.73	0.98	1.03
2.2	0.130	2331	4.30	0.94	1.14
0.3	0.300	2701	3.70	0.87	1.32
0.4	0.610	3375	2.96	0.78	1.65
0.5	1.100	4442	2.25	0.68	2.17

Taking the spring stiffness, k (load per unit deflection) of the cracked beam as the inverse of the compliance (deflection per unit load), one gets:

$$\frac{k(a)}{Et} = \frac{1}{\beta^2 (34\ \text{In} + 4\beta)} \quad (14)$$

and the ratio $k(a)/ko$ as

$$\frac{k(a)}{ko} = \frac{4\beta}{34\ \text{In} + 4\beta} \quad (15)$$

The relationship between the compliance and the crack length, and between the spring stiffness and the crack length are shown in Fig. (4) for $\beta = 8$.

4. FUNDAMENTAL FREQUENCY

The fundamental frequency, ωo, given by Eqn (1) depends on the spring stiffness, k, which is a function of a/W. The variation of $\omega(a)/\omega o$ with a/W is shown in Fig. 5. The fundamental frequency decreases with increasing of crack length.

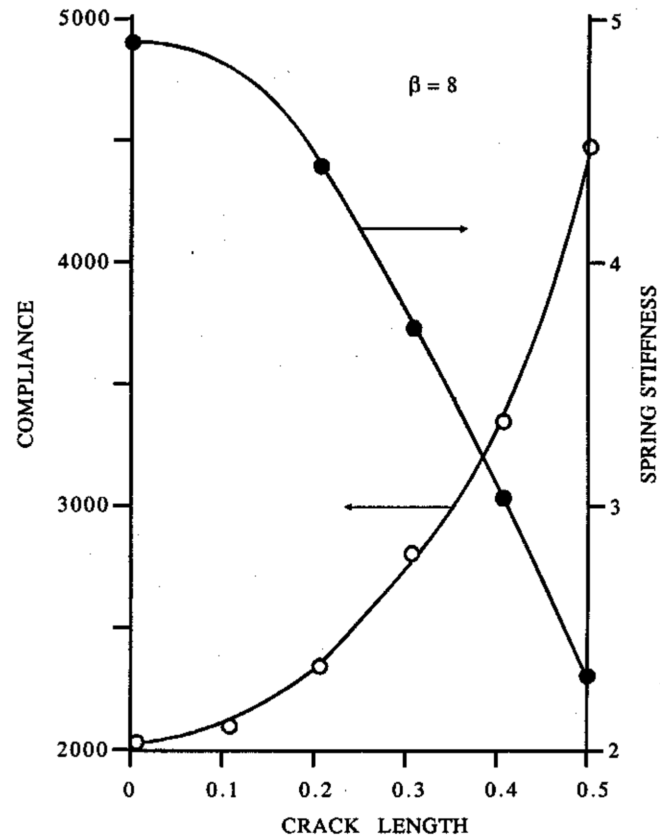

Figure 4. Relationship between compliance and crack length, and between spring stiffness and crack length.

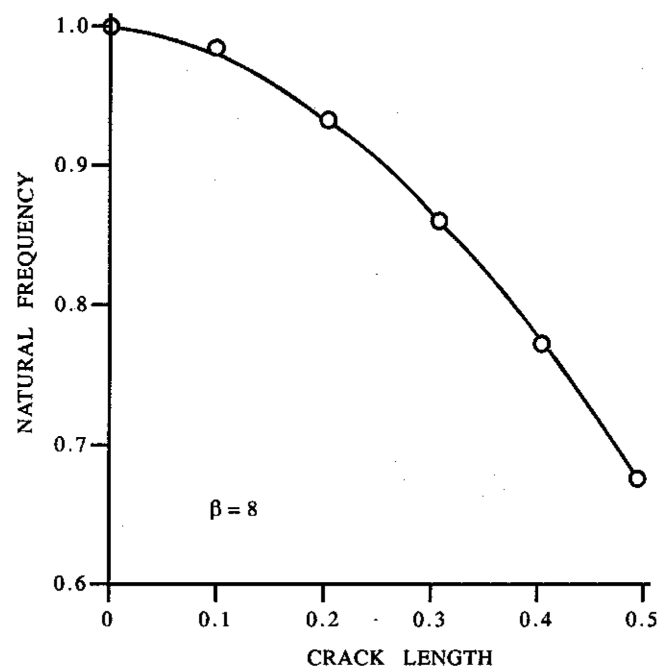

Figure 5. Relationship between natural frequency and crack length.

The increase in static deflection $Yo(a)$ with crack length, a/W, as a ratio of the static deflection at $a/W = 0$ is shown in Fig. 6.

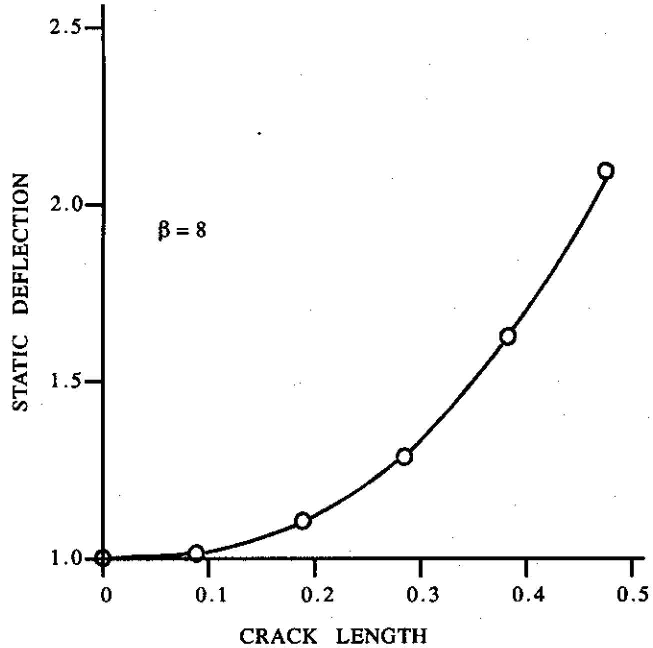

Figure 6. Increase in static deflection with crack length

5. FREE VIBRATIONS

5.1 Crack at Fixed End

In the case of a cantilever beam of elastic material with no damping, free vibration will continue with a period of oscillation τ which is inversely proportional to the frequency. Thus, the ratio of the period of oscillation, $\tau(a)$ of the beam with end crack to the period of oscillation, τo, of the beam without a crack, can be given as

$$\tau(a)/\tau o = \omega o/\omega(a) = \sqrt{(ko/k(a))} \qquad (16)$$

Figure 7 shows schematically the period of oscillations for $a/W = 0$ and $a/W = 0.4$.

5.2 Crack at Mid-span

Consider the vibration of an elastic cantilever beam with a crack at the mid-span, as shown in Fig. 8. The problem will be similar to two spring-mass system as shown in the figure. The portion of the beam nearer to the fixed end will have the

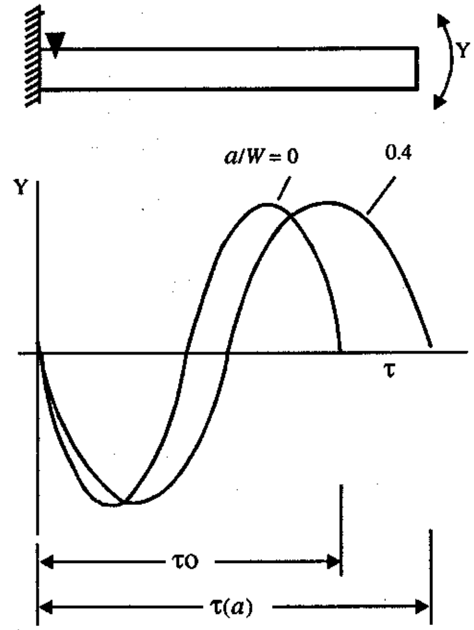

Figure 7. Effect of crack length on the period of oscillation

spring stiffness $k1$ equal to ko, whereas the portion on the other side at the free end will have the spring stiffness $k2 = k(a)$.

Thus taking the beam with the mid-span crack as two spring-mass system, the ratio of the

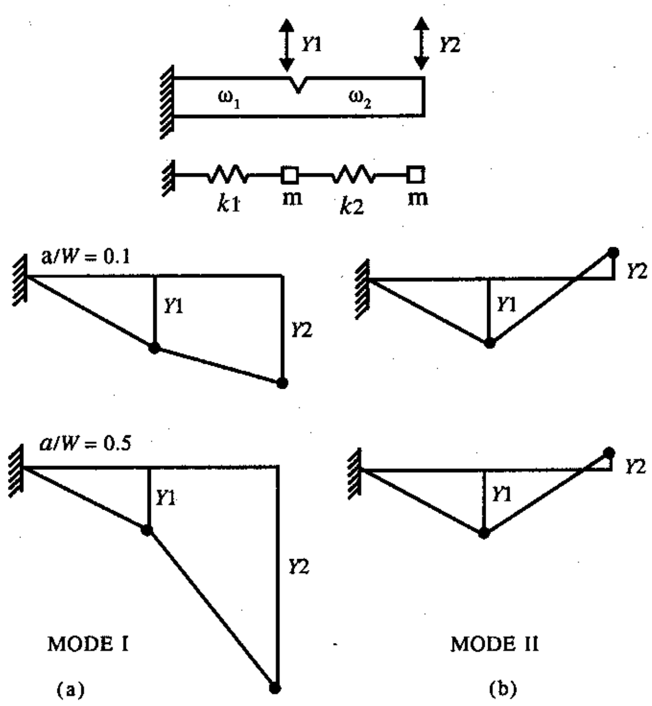

Figure 8. Two modes of vibration in beam with mid-span crack

amplitudes $Y1$ at the centre and $Y2$ at the free end is obtained as

$$\frac{Y1}{Y2}=1-\frac{m}{k2}\left[\frac{k1+2k2}{2m}+\sqrt{\left[\frac{k1+2k2}{2m}\right]^2-\frac{k1k2}{m^2}}\right] \quad (17)$$

Consider the cantilever with $L/W = \beta = 8$. For the mid-span crack with span length on either side being $L/2$, the ratio of the spring stiffness values $k1/k2$ is given as

$$k1/k2 = ko/k(a)$$

$$=\frac{\beta_1^2(34\,\mathrm{In}+4\beta_1)}{\beta_1^2(4\beta_1)}=\frac{34\,\mathrm{In}+16}{16} \quad (18)$$

where $\beta_1 = 4$. The values of the ratio $Y1/Y2$ with a/W values are given in Table 2. The maximum amplitudes in the two modes are also shown in Fig. 8 [(a) for mode I and (b) for mode II].

Table 2. Values of the amplitude ratios $Y1/Y2$ in mode I and mode II of vibration of a beam with crack at the mid-span, for $\beta = 8$.

a/W	0.100	0.200	0.30	0.40	0.50
In	0.031	0.130	0.30	0.61	1.10
$Y1/Y2$ (mode I)	0.600	0.534	0.48	0.37	0.27
$Y1/Y2$ (mode II)	–1.660	–1.820	–2.12	–2.67	–3.60

6. FORCED VIBRATIONS

6.1 Crack at Fixed End

Since, the introduction of a crack reduces the spring stiffness k and the natural frequency of the beam, beams with cracks will resonate at a lesser exciting frequency, ω, as the fundamental frequency $\omega(a)$, of the cracked beam decreases with increasing crack length. The occurrence of resonance will get shifted to a lower value of $\omega/\omega o$ (less than 1), where ω is the exciting frequency and ωo is the natural frequency of the uncracked beam.

The amplitude variation, Y®, in the normal case, can be written as

$$Y® = \frac{Yo}{1-[\omega/\omega o]^2} \quad (19)$$

In a similar way, the amplitude variation $Y(a)$® in the presence of a crack at the fixed end can be given in terms of the static deflection $Yo(a)$ as

$$Y(a)® = \frac{Yo(a)}{1-[\omega/\omega(a)]^2} \quad (20)$$

The variation of the amplitude $Y(a)$® with the exciting frequency $\omega/\omega o$, is shown in Fig. 9 for $a/W = 0.4$ and $\beta = 8$.

6.2 Crack at Mid-span

With the crack at the mid-span, the vibrations of the cantilever beam can be analysed as two spring-mass system. With the force, $Fo \sin \omega t$, acting at the free end of the beam, the amplitudes $Y1$ and $Y2$ at the mid-span and at the free end,

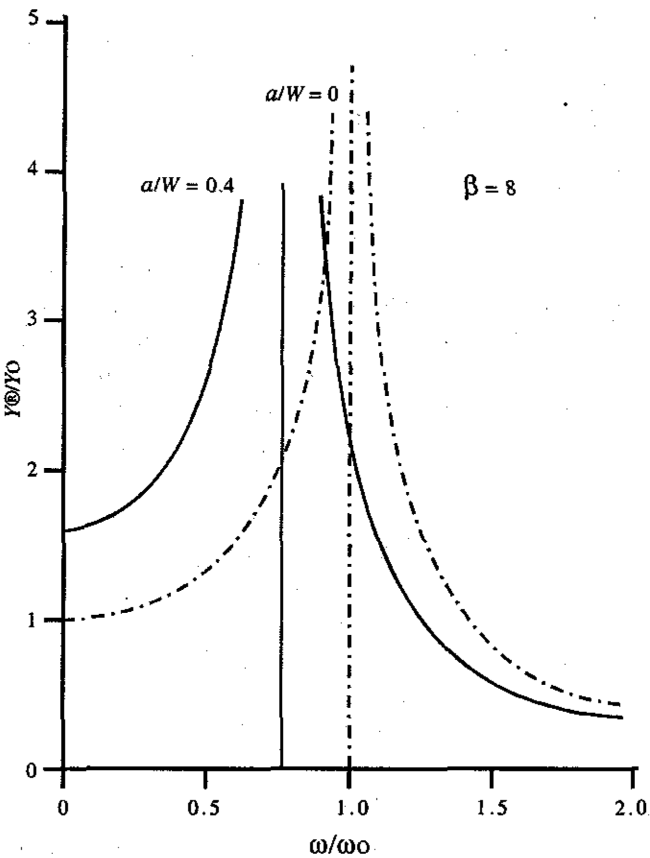

Figure 9. Occurrence of resonance in the presence of a crack at fixed end.

respectively can be written as

$$\frac{Y1}{Yst}=\frac{1}{\left[\frac{\omega^4}{\omega_1^2\omega_2^2}-\frac{\omega^2}{\omega_2^2}-\frac{2\omega^2}{\omega_1^2}\right]+1} \tag{21}$$

and

$$\frac{Y2}{Yst}=\frac{1+\frac{\omega_1^2}{\omega_2^2}-\frac{\omega^2}{\omega_2^2}}{\left[\frac{\omega^4}{\omega_1^2\omega_2^2}-\frac{\omega^2}{\omega_2^2}-\frac{2\omega^2}{\omega_1^2}\right]+1} \tag{22}$$

where $Yst = Fo/k1$. As a particular case with $a/W = 0.2$, and $\beta = 8$, one gets $k1/k2 = 1.276 = (\omega_1/\omega_2)^2$ and the two natural frequencies are, $\omega n1 = 1.49\ \omega_1$ and $\omega n2 = 0.6\ \omega_1$. Figure 10 shows the variation of the amplitudes $Y1/Yst$ and $Y2/Yst$ with the exciting frequency ω/ω_1.

7. CONCLUSIONS

The vibration behaviour of a cantilever beam with cracks at the fixed end and at the mid-span has been analysed based on the fracture mechanics concepts of crack extension force and stress intensity factor. The analysis shows that

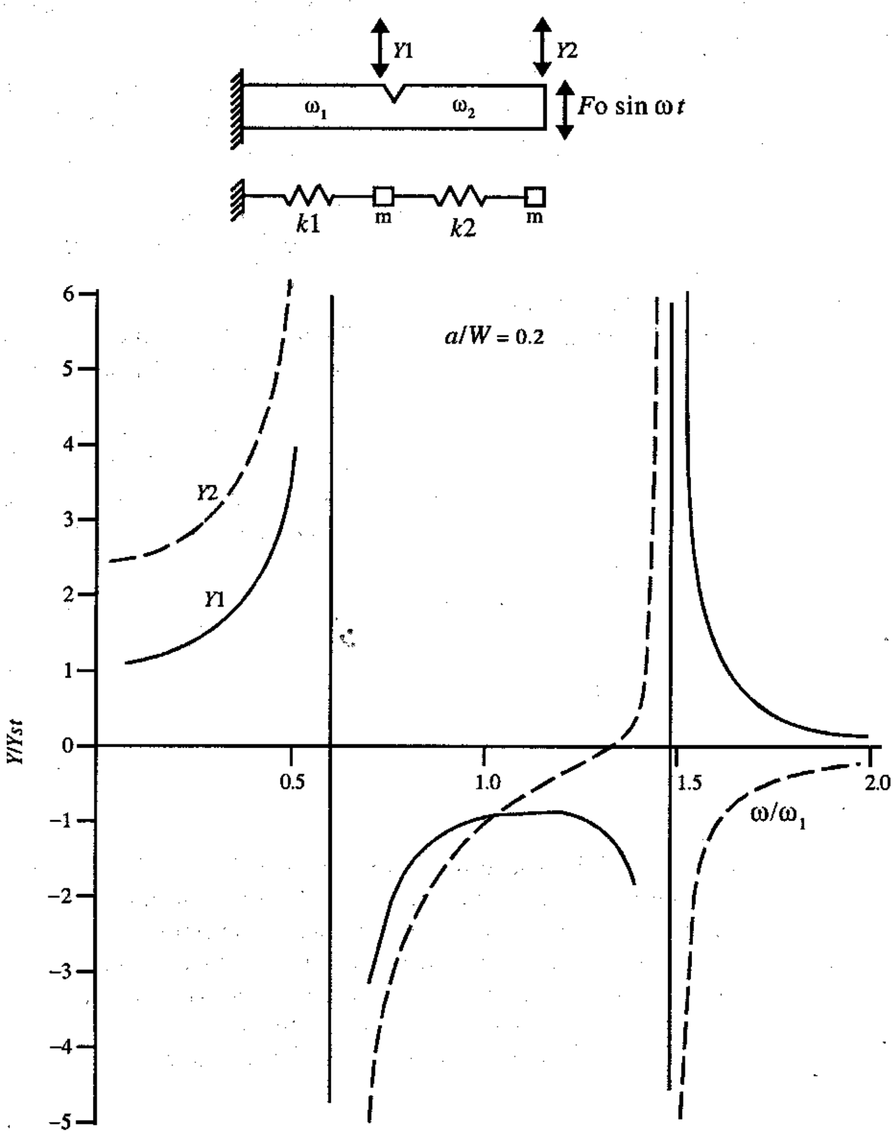

Figure 10. Variation of two amplitudes with exciting frequency

a) With increasing crack length, the spring stiffness and the natural frequency of the beam decrease.

b) In free vibration, the period of oscillation of a beam with a crack is higher than that of the beam without a crack.

c) In forced vibration of a beam with a crack at the fixed end, the occurrence of resonance gets shifted to a lower value of (ω/ω_o), where ω_o is the natural frequency of the beam without a crack.

d) In the case of a beam with a crack at the mid-span, the vibration behaviour can be analysed treating beam as two spring-mass system. The analysis yields the amplitudes in free vibration with two modes and in forced vibration, the occurrence of resonance and vibration pattern with the exciting frequency.

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Contributor

Prof (Dr) VM Radhakrishnan obtained his BE (Mech Engg) from the Madras University, MTech and PhD both from the Indian Institute of Technology (IIT) Madras, Chennai. He worked as faculty member at the IIT Madras from 1962 to 2001 and retired as Professor Emeritus in 2001. He has around 40 years of teaching and research experience in the areas of machine design, mechanics, and strength of materials. He has guided a number of MS and PhD students and published more than 250 papers in national/international journals.