Defence Science Journal, Vol. 74, No. 3, May 2024, pp. 316-320, DOI : 10.14429/dsj.74.19857

Response of a Stretched String Subjected to a Moving Mass

Kush Kumar* and Shakti S. Gupta

#Department of Mechanical Engineering, Indian Institute of Technology, Kanpur - 208 016, India
**E-mail: kushk20@iitk.ac.in*

ABSTRACT

In this paper, the dynamics of a taut horizontal string with a constant velocity moving mass including its rotary inertia is modelled. The equation of motion is solved using Galerkin's approach, employing appropriate comparison functions. A discontinuity or jump in the trajectory of the mass has been established when the mass is about to leave the string. The consideration of rotary inertia in the model does affect the spatial location of the jump in the trajectory of the moving mass.

Keywords: Moving mass; String; Rotary inertia; Galerkin method

NOMENCLATURE

ρ	: Mass density of the string, kg m^{-3}
l	: Length of the string, m
A	: Cross-section area of the string, m^2
m, M	: Dimensional (kg) and nondimensional mass of the moving mass respectively
I, P	: Imensional (kg m^2) and nondimensional mass moment of inertia of the moving mass, respectively
u, B	: Dimensional (ms^{-1}) and nondimensional velocity of the moving mass, respectively
x_c, X_c	: Dimensional (m) and nondimensional position of the moving mass, respectively
x	: Position of a point on the string, m
$w(x,t)$	: Dimensional (m) and nondimensional transverse displacement of the string
t, $\bar{t}$	: Dimensional (s) and nondimensional time, respectively
t_{tot}	: Time taken by the moving mass to cover the string length, s
T_c,T	: Dimensional (kg ms^{-2}) and nondimensional constant tension in the string, respectively
M, C, K	: $n\times n$ mass, damping and stiffness matrices of the system, respectively
f	: $n\times 1$ force vector acting on the system
q	: $n\times 1$ generalized coordinate variable vector
W_s	: s^{th} comparison function in Galerkin formulation
ϕ_s	: $s\,\pi$, s=1,2,…,n

1. INTRODUCTION

The dynamic response of strings or cables subjected to moving loads has been extensively studied. Some of its engineering applications are cable cars, elevator cables, helicopters with a hanging load, guide ways in robotic systems, deep mine hoisting cables, and mooring cables. There can be three different classes of a moving load on a string: a moving force[1], a moving mass[2-3] or a moving oscillator[4-5]

Fryba[6] has comprehensively compiled the work on problems concerning structures subjected to moving loads. Bajer and Dyniewicz[7] have numerically addressed several structural problems involving moving inertial forces. To answer the problems, they used the space–time finite element approach. They also mention Renaudot's Approach[7] and Yakushev's Approach[7-8] to tackle the inertia effect of the moving mass on a beam. Ouyang[9] has recently published a tutorial on moving-load dynamic problems.

Al-Qassab and Nair[10] formulated the free vibration of a catenary cable subjected to an attached mass using Hamilton's principle and Galerkin's solution. Further, they used Fourier and wavelet transformations to obtain the natural frequencies and mode shapes. Sofi & Muscolino[4] analysed the dynamics of a suspended cable carrying moving oscillators, while Ghadiri and Kazemi[5] analysed the nonlinear dynamics of a suspended cable carrying a moving mass-spring-damper system.

Smith[2] and Rodeman[11], *et al.* addressed the linear response of a taut string subjected to a moving mass with uniform and accelerating motion, respectively. Ferretti[12], *et al.* and Luongo[13], *et al.* conducted investigations on a taut string with a moving mass and a moving train of forces, respectively. Ferretti[1], *et al.* studied the response of a taut string to a moving force. Here, the authors have included the geometric nonlinearity of the deforming string, considering the quasi-static stretch and Kirchhoff strain model. The same model has been used by Ferretti[14], *et al.* to model a horizontally taut and geometrically non-linear string subjected to a force-driven point mass.

Over a century ago, Stokes[15] studied the problem of a heavy inertial mass particle moving along a bridge. A mass less Euler-Bernoulli beam was employed to model the bridge.

Received : 08 February 2024, Revised : 10 February 2024
Accepted : 19 February 2024, Online published : 10 May 2024

Stokes[15] demonstrated that the mass particle's vertical displacement at the bridge's end is usually not zero, implying that the trajectory of the mass particle going over the bridge is discontinuous. This discontinuity was referred to by many authors as a paradox in the trajectory of the moving mass. It has been argued that this problem arises because the beam's inertia is ignored. Similarly, in the case of a taut horizontal string, Smith[2] presented an explicit and exact solution for a taut string subjected to a moving mass particle and showed the condition for a jump in the trajectory. For an inertial string, he used the wave propagation solution and concluded that the growth of kinks gives rise to this jump; for a non-inertial string, he transformed the dynamics equation for displacement of the moving mass and mathematically derived the condition for the jump. Dyniewicz and Bajer[3] studied and proved the paradoxical trajectory of a mass moving on a taut string near the end, using the series solution for the trajectory given Fryba[6] Gavrilov[16], *et al.* revisited this paradox in the trajectory of a mass particle moving on a taut string for a more generalised string-moving mass system. In their model, the ends of the string were hinged to vertical springs. They derived the equation of motion for an extended nonlinear model of the string-moving mass system and considered wave pressure force for the conservative system to get the lateral and longitudinal displacement of the moving mass for a small strain in the string.

In all the studies done so far related to the paradox in the trajectory of a moving mass on a string, none of them have considered the rotary inertia of the mass. In this work, the same has been included. Further, its effect on the jump in the trajectory of the moving mass is discussed.

This paper is organised as follows: in Sec. 2, we present the mathematical model and assumptions used in this study. The equation of motion is then derived using Hamilton's principle. The equation of motion is then non-dimensionalized. For a particular initial condition, we solved it using the Galerkin method, which converts the equation of motion into the reduced matrix form and is then solved using the Runge-Kutta method. In Sec. 3, we go over the results, including the paradox in the trajectory of the moving mass. And in Sec. 4, we conclude our study.

2. MATHEMATICAL FORMULATION AND GOVERNING EQUATION

Consider the string-moving mass system as shown in Fig. 1. The string has a mass density ρ, length l, and cross-section area A. The moving mass has a mass m, mass moment of inertia I about the y-axis about its centre of mass. At any instant of time, the moving mass has a velocity u. The acceleration due to gravity is g acting in the downward direction. The string aligns along the x-axis, and the origin O lies at the left fixed end of the string. From this end, the mass starts moving at t=0 with a constant velocity of u and reaches the right end or the terminating end of the string at time $t=t_{tot}$. We focus on the dynamics of the string until the mass reaches the terminating end. After this, the string vibrates freely. The transverse displacement $w(x,t)$ of the material points of the string is measured along the z-axis from their mean position. It has been assumed that the transverse displacement of the string is small compared to its length and that it is inextensible. Also, the tension T in the string is high enough that the moving mass does not change its value. There is no damping present in the system. The contact between the moving mass and the string is assumed to be a point contact. The dimension of the moving mass is small compared to the length and thickness/diameter of the string; however, it possesses some finite radius of gyration r such that $I=mr^2$.

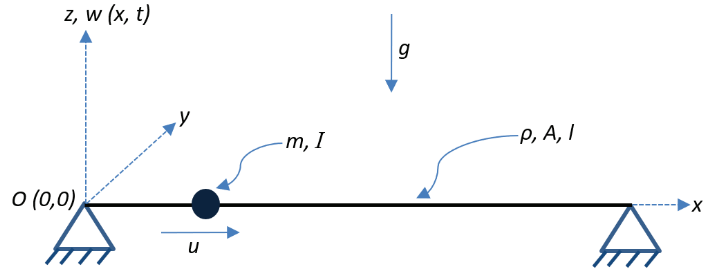

Figure 1. A moving mass on a taut horizontal string.

The rotary inertia of the moving mass cannot be neglected even for small deformation of the string, because the rate of change of the slope at the location of the moving mass might impart the rotational kinetic energy to the mass, which could be comparable to other components of the kinetic energy of the system. Figure 2(a) shows the orientation of the moving mass at time t and the small angle through which it has been rotated is given by $\theta(x,t)|_{x=x_c}$, where $x_c=ut$ is the current position of the moving mass along x-axis from the origin. Figure 2(b) shows the orientation when the rotary inertia of the moving mass is neglected.

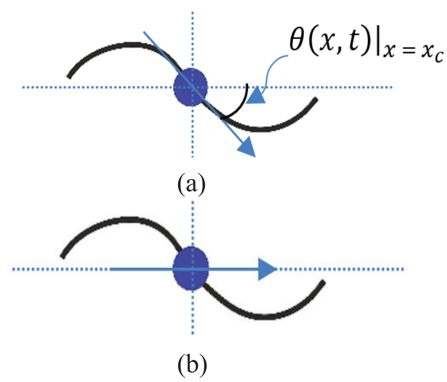

Figure 2. Rotary inertia of the moving mass, (a) orientation of the moving mass considering rotary inertia (b) orientation of the moving mass without rotary inertia.

2.1 Equation of Motion

The equation of motion of the system is derived using Hamilton's principle[17]. Towards this the system's total potential and kinetic energies are written as follows:

$$U_{string} = \int_0^l \frac{T}{2}\left(\frac{dw(x,t)}{dx}\right)^2 dx,$$

$$T_{string} = \int_0^l \frac{\rho\, A}{2}\left(\frac{dw(x,t)}{dt}\right)^2 dx,$$

$$T_{mass} = \frac{m}{2}u^2 + \frac{m}{2}\left(\frac{d}{dt}\left(w(x,t)|_{x=x_c}\right)\right)^2 + \frac{I}{2}\left(\frac{d}{dt}\left(\theta(x,t)|_{x=x_c}\right)\right)^2.$$

Here, U_{string} is the potential energy of the string. T_{string} is the kinetic energy of the string. T_{mass} is the total kinetic energy of the mass. In T_{mass}, the first and the second terms correspond to the kinetic energy due to translation motion along x-axis and z-axis respectively. In T_{mass}, the third term corresponds to the kinetic energy due to rotation of the mass. And the virtual work done by gravity on a moving mass, is given by:

$$\delta W = -\int_0^l \big(mg\delta(x-x_c)\big)\delta w(x,t)\ dx.$$

After substituting the above energy and the virtual work done expressions in the following Eqn.

$$\delta\int_{t_1}^{t_2}\left(T_{string}+T_{mass}-U_{string}\right)dt+\delta\int_{t_1}^{t_2}W\,dt=0. \quad (1)$$

we obtain the following Eqn. of motion for the string:

$$\rho A\frac{\partial^2 w(x,t)}{\partial t^2}-T_c\frac{\partial^2 w(x,t)}{\partial x^2}+mg\delta(x-x_c)+ m\frac{d^2 w(x_c,t)}{dt^2}\delta(x-x_c)- I\left\{\frac{d^2}{dt^2}\left(\frac{\partial w(x,t)}{\partial x}\Big|_{x=x_c}\right)\right\}\frac{\partial\delta(x-x_c)}{\partial x}=0 \quad (2)$$

for the response of the string. In Eqn. (2) $\delta(x-x_c)$ is the Dirac delta function and $\frac{\partial\delta(x-x_c)}{\partial x}$ is its partial derivative with respect to x, At this juncture, it is worth mentioning that in deriving the Eqn. of motion employing Newton-Euler mechanics for the same system with a non-rotating mass moving with a constant velocity, Yakushev[8] wrote the rate of change of linear momentum for the moving mass in the following form:

$$\frac{d}{dt}\left[\delta(x-ut)m\frac{dw(ut,t)}{dt}\right]=-\delta'(x-ut)mu\frac{dw(ut,t)}{dt}+ \delta(x-ut)m\frac{d^2w(ut,t)}{dt^2}, \quad (3)$$

However, in the equation of motion (see Eqn. (2)) derived here using Hamilton's principle, we would obtain only the second term of Eqn. (3) for the mass moving with a constant velocity. It has been shown by Langer and Klasztorny[18] that the term $-\delta'(x-ut)mu\frac{dw(ut,t)}{dt}$ in Eqn. (3) gets cancelled with an opposite dipole of the same magnitude acting at the same point along the z-axis.

2.2 Non-Dimensional Equation and Solution Procedure

To get the non-dimensional form of the Eqn. of motion, we introduce the following non-dimensional quantities:

$$\bar{x}=\frac{x}{l},\quad \bar{t}=t\sqrt{\frac{g}{l}},\quad \bar{w}(\bar{x},\bar{t})=\frac{w(x,t)}{l},$$

$$T=\frac{T_c}{\rho Alg},\quad M=\frac{m}{\rho Al},$$

$$B=\frac{u}{\sqrt{gl}},\quad P=\frac{I}{\rho Al^3}\quad \text{and,}\quad X_c=B\bar{t}.$$

Substituting these values in Eqn.(2), we get the non-dimensional form of the Eqn. of motion as

$$\frac{\partial^2 w(x,t)}{\partial t^2}-T\frac{\partial^2 w(x,t)}{\partial x^2}+M\delta(x-X_c)+ M\frac{d^2 w(X_c,t)}{dt^2}\delta(x-X_c)- P\left\{\frac{d^2}{dt^2}\left(\frac{\partial w(x,t)}{\partial x}\Big|_{x=X_c}\right)\right\}\frac{\partial\delta(x-X_c)}{\partial x}=0. \quad (4)$$

Overbars from w, x and t are removed henceforth for the sake of simplicity.

2.3 Solution Procedure

The Eqn. of motion is solved using the Galerkin method. Towards this, we write the transverse deflection $w(x,t)$ as

$$w(x,t)=\sum_{s=1}^{n}W_s(x)q_s(t), \quad (5)$$

where, $W_s(x)$ are the comparison functions defined in $W_s(x)=sin(s\pi x)$ and $q_s(t)$ are the generalised coordinates, and n is the number of comparison functions included in the solution. The boundary conditions for the fixed ends are $w(0,t)=w(1,t)=0$.

Next, taking the inner product of equation of motion (4) with $W_r(x)$ and substituting the value of $w(x,t)$ from Eqn. (5), we obtain the discredited Eqn. of motion as

$$\mathrm{M}\ddot{\mathrm{q}}+\mathrm{C}\dot{\mathrm{q}}+\mathrm{K}\mathrm{q}=\mathrm{f} \quad (6)$$

where, components of mass matrix M, C, K and vector **f** are given, respectively, as follows.

$$M_{rs}=M\big(sin(\phi_r X_c)\,sin(\phi_s X_c)\big)+ P\big(\phi_r\phi_s\,cos(\phi_r X_c)\,cos(\phi_s X_c)\big)+\tfrac{1}{2}\delta_{rs}, \quad (7(a))$$

$$C_{rs}=2\phi_s M\big(\dot{X}_c\,sin(\phi_r X_c)\,cos(\phi_s X_c)\big)- 2P\phi_r\phi_s\big(\dot{X}_c\,cos(\phi_r X_c)\,sin(\phi_s X_c)\big), \quad (7(b))$$

$$K_{rs}=-M\left(\phi_s^2\dot{X}_c^{\,2}sin(\phi_r X_c)\,sin(\phi_s X_c)\right)- P\left(\phi_r\phi_s^3\dot{X}_c^{\,2}cos(\phi_r X_c)\,cos(\phi_s X_c)\right)-+\frac{T\phi_r^2}{2}\delta_{rs}, \quad (7(c))$$

and,

$$F_r=-M\,sin(\phi_r X_c). \quad (7(d))$$

In Eqs. (7)

$\phi_s=s\pi$, $\phi_r=r\pi$, $\dot{X}_c=\frac{dX_c}{dt}=B$, and δ_{rs} represents the Kronecker delta operator. Further, in Eqn. (6), **q** is the vector of generalised coordinates.

3. RESULTS AND DISCUSSION

The linear and coupled ordinary differential equations Eqn. (6) for the motion of the string with a moving mass are solved numerically using the Runge-Kutta method employing the *ode45* function of MATLAB[19].Compared to other numerical integration methods such as Euler's method, Simpson's method, and Picard's method, the Runge-Kutta method typically offers higher-order accuracy, which is $O(h^4)$, where h is the step size. The Taylor series method can offer even higher accuracy, but that requires the computation of higher derivatives, which is computationally expensive. Hence, we have used the Runge-Kutta method, which is comparatively easier to implement

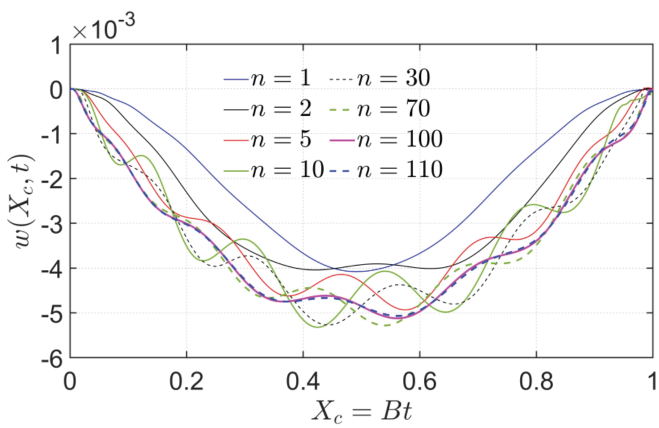

Figure 3. The convergence of the trajectory of the moving mass for B = 0.01.

than more intricate methods like the Crank-Nicolson method and Taylor series method[20]. The initial conditions are: $w(x,0)=0$ and $\dot{w}(x,0)=0$ and initially the mass is at the origin, moving towards the positive x-axis with a constant velocity u.

Unless stated otherwise, the values of the parameters used to study the dynamics are: ρ=150 kg m^{-3}, A=0.0001 m^2, T_C=1500 N, l=5 m, m=3 kg, r=0.25 m and g=9.81 m s^{-2}. The convergence of the trajectory of the moving mass is shown in Fig. 3 for B=0.01. Accordingly, n=110 is chosen for the studies. However, it is found that the number of comparison functions needed for convergence increases with the speed of the moving mass.

Figure 4 shows the trajectories of the moving mass at different non-dimensional velocities (B). It is observed that for small B(=0.45), the trajectory is symmetric about the centre, which is X_c=0.5. However, when the value of B increases, the maximum depression in the trajectory occurs after the central line towards the terminating end. Also, for a large value of B, the trajectory paradox[3] is observed, which is the jump in the trajectory (h_1, h_2, h_3) at the terminating end. The paradox exists even for lower values of B, but is negligible compared to the displacements of the moving mass at other locations.

When the value of B is larger (≥ 3.16), the magnitude of the jump (h) is comparable to the displacements of the moving mass at other moments. When B is very large (≥ 31.60), the trajectory will appear to coincide with the initial orientation of

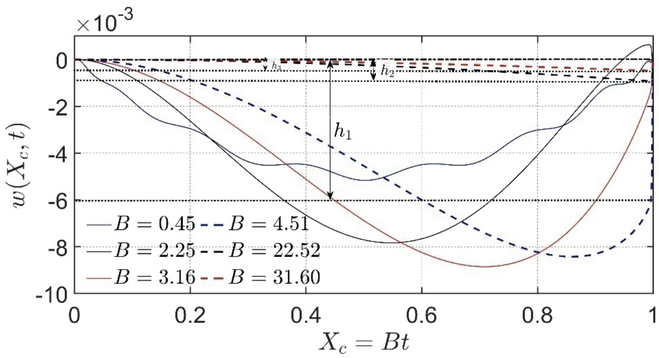

Figure 4. Trajectories of the moving mass for different non-dimensional velocities (*B*).

the string as the moving mass just traces the string without any appreciable transverse displacement in a short time.

We plotted Fig. 5 for B=4.51 to show the jump in the path of the moving mass with and without rotary inertia (P=0) and how adding more and more comparison functions leads to convergence. As n increases, the jump occurs in fewer time steps, and the trajectory becomes steeper near the terminating end. The jump lengths (h) have been found to remain constant regardless of whether rotatory inertia is taken into account or not. The trajectory when rotary inertia is considered is found to converge faster compared to when it is ignored. Also, it can be noticed that the jump in the trajectory advances by a value of AB=6.04×10^{-4}, which cannot be ignored for a jump of order h=6×10^{-3}.

4. CONCLUSION

Using Hamilton's principle, the equation of motion of a taut string with a uniformly moving mass, including its rotary inertia, is derived. The obtained equation of motion is solved using Galerkin's approach. The trajectories of the moving mass for various values of its speed are plotted. These plots reveal a jump when the moving mass reaches the terminating end of the string. This jump remains unaffected by the rotary inertia of the mass. However, including rotary inertia advances the occurrence of the jump compared to when it is neglected.

REFERENCES

1. Ferretti, M.; Piccardo, G. & Luongo, A. Weakly nonlinear dynamics of taut strings travelled by a single moving force. *Meccanica,* 2017, **52**(13), 3087–3099.
2. Smith, C.E. Motions of a stretched string carrying a moving mass particle. *J. Appl. Mech*., 1964, **31**(1), 29–37.
3. Dyniewicz, B. & Bajer, C.I. Paradox of a particle's trajectory moving on a string. *Archive of App. Mech.,* 2009*,* **79**(3), 213–223.
4. Sofi, A. & Muscolino, G. Dynamic analysis of suspended cables carrying moving oscillators. *Int. J. Solids Struct.* 2007*,* **44**(21), 6725–6743.
5. Ghadiri, M. & Kazemi, M. Nonlinear vibration analysis of a cable carrying moving mass-spring-damper. *Int. J.*

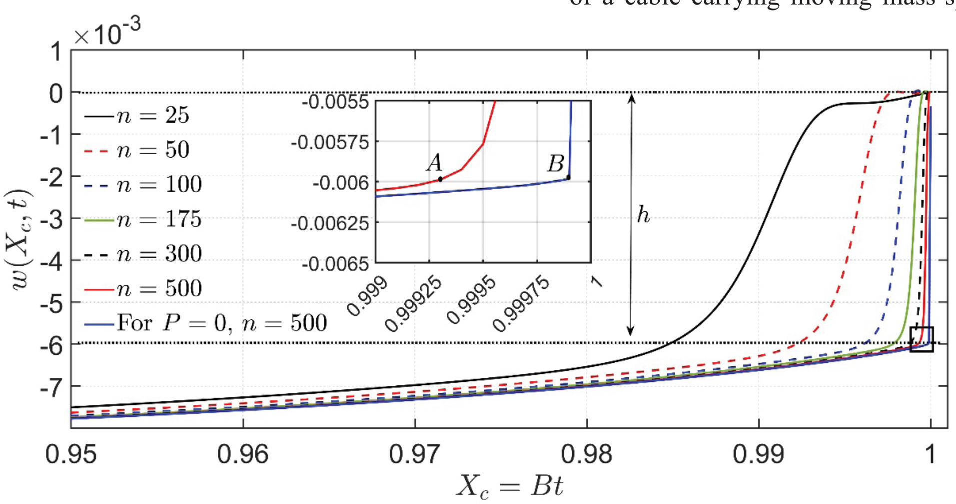

Figure 5. Convergence of the paradox trajectory near the terminating end for non-dimensional velocity *B*=4.51 and comparison with the paradox trajectory for a moving particle (*P*=0).The inset shows the trajectories near the terminating end, and the length of *AB*=6.04×10^{-4} quantifies the advances of the jump occurrence.

Stru. Stab. Dyna., 2018, **18**(02),1850030.
6. Fryba, L. Vibration of solids and structures under moving loads. Vol. 1, Springer Science & Business Media, 2013.
7. Bajer, C.I. & Dyniewicz, B. Numerical analysis of vibrations of structures under moving inertial load. Vol. 65, Springer Science & Business Media, 2012.
8. Yakushev, N.Z. Certain problems of dynamics of the beam under moving load (in Russian). Studies in plates and shells theory. 1974, **12**, 199–220.
9. Ouyang, H. Moving-load dynamic problems: A tutorial (with a brief overview). *Mech. Syst. Signal Process.*, 2011, **25** (6), 2039–2060.
10. Al-Qassab, M. & Nair, S. Wavelet-galerkin method for the free vibrations of an elastic cable carrying an attached mass. *J. Sound and Vibration*, 2004, **270**(1-2), 191–206.
11. Rodeman, R.; Longcope, D.B. & Shampine, L.F. Response of a string to an accelerating mass. *J. Appl. Mech*., 1976, 675-680.
12. Ferretti, M. & Piccardo, G. Dynamic modeling of taut strings carrying a traveling mass. *Continuum Mech. Thermodynamics,*2013*,* **25**(2), 469–488.
13. Luongo, A. & Piccardo, G. Dynamics of taut strings traveled by train of forces. *Continuum Mech. Thermodynamics*, 2016, **28**(1), 603–616.
14. Ferretti; M.; Gavrilov, S.; Eremeyev, V. & Luongo, A. Nonlinear planar modeling of massive taut strings travelled by a force-driven point-mass. *Nonlinear Dynamics,* 2019*,* **97**(4), 2201–2218.
15. Stokes, S.G.G. Discussion of a differential equation relating to the breaking of railway bridges. Printed at the Pitt Press by John W. Parker. 1849, **8**,707–735.
16. Gavrilov, S.; Eremeyev, V.G.; Piccardo, A. & Luongo, A. revisitation of the paradox of discontinuous trajectory for a mass particle moving on a taut string. *Nonlinear Dynamics,* 2016, **86**(4), 2245–2260.
17. Hamilton,W.R. Second essay on a general method in dynamics. *Philos. Trans R. Soc. Lond*., 1835, **125**, 95-144.
18. Langer, J. & Klasztorny, M. Interpretation of jakushev's description of concentrated mass moving along euler-bernoulli beam. *Archives of Civil Eng.*, 1995, **41**(1), 5–11.
19. Mathworks, MATLAB: The Language of Technical Computing: Version 5, MathWorks, 1984.
20. Chapra, S.C. & Canale, R.P. Numerical Methods for Engineers, McGraw-Hill, New York, 2012.

CONTRIBUTORS

Mr Kush Kumar obtained his Bachelor of Technology degree in Mechanical Engineering from the Malaviya National Institute of Technology Jaipur (MNIT Jaipur), India. His research interests include moving loads on structures, the nonlinear finite element method (FEM) and nodal position FEM.
His contributions to the present research work are: Conceptualisation of problem definition, literature survey, analysis of results and writing manuscript.

Prof Shakti S. Gupta obtained his PhD at Virginia Tech in the United States and currently holds the position of a Professor at the Indian Institute of Technology, Kanpur, India (IIT Kanpur). His areas of research are linear and nonlinear structural mechanics; and the mechanics of nano-materials and their characterization using molecular simulations.
He reviewed the present work, validated the results, and from time to time provided tangible inputs to improve the research.