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# On the Class of Exact Solutions of an Incompressible Second-order Fluid Flow by Creating Sinusoidal Disturbances

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**ABSTRACT**

Exact solution of an incompressible fluid of second-order by causing disturbances in the liquid, which is initially at rest due to bottom oscillating sinusoidally, has been obtained in this study. The results presented are in terms of nondimensional elasticoviscosity parameter ($\beta$) which depends on the non-Newtonian coefficient and the frequency of excitation of the external disturbance while considering the porosity ($K$) of the medium. The flow parameters are found to be identical with that of Newtonian case as $\beta \to 0$ and $K \to \infty$.

**Keywords:** Elasticoviscous fluid, second-order fluid, retarded history, porous media, fluid flow, heat transfer

## NOMENCLATURE

$g(s)$ Given history

$g_\psi(s)$ Retarded history

$\psi$ Retardation factor

$S$ Stress tensor

$P$ Indeterminate hydrostatic pressure

$U_i$ Velocity component in the $i^{th}$ direction

$A_i$ Acceleration component in the $i^{th}$ coordinate

$T$ Time

$\Phi_1$ Coefficient of viscosity

$\Phi_2$ Coefficient of elasticoviscosity

$\Phi_3$ Coefficient of cross viscosity

$u_i$ Nondimensional velocity component along the $i^{th}$ coordinate

$\rho$ Density of the fluid

$\beta$ Nondimensional elasticoviscosity

$p$ Nondimensionalised indeterminate hydrostatic pressure

$a_i$ Nondimensionalised acceleration component in the $i^{th}$ direction

$L$ Characteristic length

$K$ Nondimensionalised porosity factor

$\nu_c$ Nondimensionalised cross viscosity parameter

$\sigma$ Frequency of excitation

$k$ Permeability of the material

## 1. INTRODUCTION

Viscous fluid flow over wavy wall has attracted the attention of relatively few researchers although

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the analysis of such flows finds applications in different areas such as transpiration cooling of reentry vehicles and rocket boosters, crosshatching on ablative surfaces and film vaporisation in combustion chambers. Especially, where the heat and mass transfer takes place in the chemical processing industry, the problem, by considering the permeability of the bounding surface in the reactors, assumes greater significance.

In view of several industrial and technological applications of this technology, Ramacharyulu[1] studied the problem of the exact solutions of two-dimensional flows of a second-order incompressible fluid considering the rigid boundaries. Later Lekoudis[2], *et al.* presented a linear analysis of the compressible boundary layer flow over a wall. Subsequently, Shankar and Sinha [3] studied the problem of Rayleigh over a wavy wall. The effect of small amplitude wall waviness upon the stability of the laminar boundary layer was studied by Lessen and Gangwani[4]. Further, the problem of free-convective heat transfer in a viscous incompressible fluid confined between a vertical wavy wall and a particle flat wall was examined by Vajravelu and Shastri[5], and thereafter, by Das and Ahmed[6].

The free-convective flow of a viscous incompressible fluid in a porous medium between the two long vertical wavy walls was investigated by Patider and Purohit [7]. Taneja and Jain[8] examined the problem of MHD flow with slip effects and temperature-dependent heat source in a viscous incompressible fluid confined between a long vertical wall and a parallel flat plate.

In all the above investigations, the fluid under consideration was viscous, incompressible and one of the bounding surfaces had a wavy character. The present analysis is aimed to examine the nature of the fluid flow considering an additional property, namely elasticoviscosity and also by creating sinusoidal disturbance at the bottom while the fluid is resting on the plate.

## 2. MATHEMATICAL FORMULATION OF THE PROBLEM

Noll[9] defined a simple material as a substance for which stress can be determined with complete knowledge of the history of the strain. It is called a simple fluid, if it has the property that all the local states, with the same mass density, are intrinsically equal in response, with all observable differences in response being due to definite differences in the history. For any given history $g(s)$, a retarded history $g_\psi(s)$ can be defined as

$$g_\psi(s)=g(\psi s) \qquad 0 \le s \le \infty \qquad 0 \le \psi \le 1 \tag{1}$$

$\psi$ being termed as a retardation factor. Assuming that the stress is more sensitive to recent deformation, than to the deformations at distant past, Coleman and Noll[10] proved that the theory of simple fluids yields the theory of perfect fluids as $\psi \to 0$ and that of Newtonian fluids as a correction (up to the order of $\psi$) to the theory of the perfect fluids. Neglecting all the terms of the order of higher than two in $\psi$, one has incompressible elasticoviscous fluid of second-order, whose constitutive relation is governed by

$$S = -PI + \phi_1 E^{(1)} + \phi_2 E^{(2)} + \phi_3 E^{(1)^2} \tag{2}$$

where $\quad E^1_{ij} = U_{i,j} + U_{j,i} \tag{3}$

and $\quad E^2_{ij} = A_{i,j} + A_{j,i} + 2U_{m,i} U_{m,j} \tag{4}$

In the above equations, $S$ is the stress tensor, $U_i$ and $A_i$ are the components of velocity and acceleration in the direction of the $i^{th}$ coordinate $X$, while $P$ is the indeterminate hydrostatic pressure. The coefficients, $\Phi_1, \Phi_2$, and $\Phi_3$ are the material constants. The constitutive relation for general Rivlin and Ericksen[11] fluid also reduces to Eqn (2) when the squares and higher orders of $E^2$ are neglected, while the coefficients being constants. Also the non-Newtonian models considered by Reiner[12] could be obtained from Eqn (2) when $\Phi_2$=0 and naming

$\Phi_2$ as the coefficient of cross viscosity. With reference to the Rivlin and Ericksen fluids, $\Phi_2$ may be called the coefficient of viscosity.

It has been reported that a solution of polyisobutylene in cetane behaves as a second-order fluid and that Markovitz determined the constants, $\Phi_1, \Phi_2$, and $\Phi_3$. In many of the chemical processing industries, slurry adheres to the reactor vessels and gets consolidated. As a result, the chemical compounds within the reactor vessel percolate through the boundaries, causing loss of production and also consume more reaction time. In view of such technological and industrial importance of the phenomena, wherein the heat and mass transfer takes place in the chemical industry, the problem of the permeability of the bounding surfaces in the reactors attracts the attention of several investigators.

The present study aims at finding a class of exact solutions for the flow of incompressible, second-order fluid, considering the porosity factor of the bounding surfaces and compare the results with those in the Newtonian case. The disturbance due to sinusoidal oscillation of the bottom of a semi-infinite depth was studied. The results have been expressed in terms of a nondimensional porosity parameter $K$, which depends on the non-Newtonian coefficient, $\Phi_2$ and the frequency of excitation, $\sigma$. It is noticed that the flow properties are identical with those in the Newtonian case ($K = \infty$).

If $V$ ($U_1$, $U_2$, $U_3$) is the velocity component and $F$ ($F_X$, $F_Y$, $F_Z$) is the body force acting on the system, then the equations of motion in $X$, $Y$ and $Z$ directions are given by

$$\rho \frac{DU_1}{DT} = \rho F_X + \frac{\partial S_{XX}}{\partial X} + \frac{\partial S_{XY}}{\partial Y} + \frac{\partial S_{XZ}}{\partial Z} \quad (5)$$

$$\rho \frac{DU_2}{DT} = \rho F_Y + \frac{\partial S_{YX}}{\partial X} + \frac{\partial S_{YY}}{\partial Y} + \frac{\partial S_{YZ}}{\partial Z} \quad (6)$$

$$\rho \frac{DU_3}{DT} = \rho F_Z + \frac{\partial S_{ZX}}{\partial X} + \frac{\partial S_{ZY}}{\partial Y} + \frac{\partial S_{ZZ}}{\partial Z} \quad (7)$$

where $\frac{D}{DT} = \frac{\partial V}{\partial T} + V.\nabla V$

If the bounding surface is porous, then the rate of percolation of the fluid is directly proportional to the cross-sectional area of the filter bed and the total force, say the sum of the pressure gradient and the gravity force. In the sense of Darcy, the flux of the fluid ($q$) is:

$$q = CA\left(\frac{P_1 - P_2}{H_1 - H_2} + \rho G\right) \quad (8)$$

where $A$ is the cross-sectional area of the filter bed, $C = k/\mu$, in which $k$ is the permeability of the material and $\mu$ is the coefficient of viscosity, and $q$ is the flux of the fluid. A straightforward generalisation of the Eqn (8) yields:

$$V = -\frac{k}{\mu}[\nabla P + \rho G \eta] \quad (9)$$

where $V$ is the velocity vector and $\eta$ is the unit vector along the gravitational force taken in the negative direction. If other external forces are acting on the system, instead of gravitational force, then the velocity vector is:

$$V = -\frac{k}{\mu}[\nabla P - \rho F] \quad (10)$$

In the absence of external forces, the velocity vector becomes:

$$V = -\frac{k}{\mu}\nabla P$$

which gives $\nabla P = -\frac{\mu}{k} V$

Therefore, the net resulting equations of motion in the $X$, $Y$, and $Z$ directions are:

$$\rho \frac{DU_1}{DT} = \rho F_X + \frac{\partial S_{XX}}{\partial X} + \frac{\partial S_{XY}}{\partial Y} + \frac{\partial S_{XZ}}{\partial Z} - \frac{\mu}{k} U_1 \quad (11)$$

$$\rho \frac{DU_2}{DT} = \rho F_Y + \frac{\partial S_{YX}}{\partial X} + \frac{\partial S_{YY}}{\partial Y} + \frac{\partial S_{YZ}}{\partial Z} - \frac{\mu}{k} U_2 \quad (12)$$

$$\rho \frac{DU_3}{DT} = \rho F_Z + \frac{\partial S_{ZX}}{\partial X} + \frac{\partial S_{ZY}}{\partial Y} + \frac{\partial S_{ZZ}}{\partial Z} - \frac{\mu}{k} U_3 \quad (13)$$

In the absence of the external body forces, the Eqns (11)-(13) reduce to

$$\rho \frac{DU_1}{DT} = \frac{\partial S_{XX}}{\partial X} + \frac{\partial S_{XY}}{\partial Y} + \frac{\partial S_{XZ}}{\partial Z} - \frac{\mu}{k} U_1 \quad (14)$$

$$\rho \frac{DU_2}{DT} = \frac{\partial S_{YX}}{\partial X} + \frac{\partial S_{YY}}{\partial Y} + \frac{\partial S_{YZ}}{\partial Z} - \frac{\mu}{k} U_2 \quad (15)$$

$$\rho \frac{DU_3}{DT} = \frac{\partial S_{ZX}}{\partial X} + \frac{\partial S_{ZY}}{\partial Y} + \frac{\partial S_{ZZ}}{\partial Z} - \frac{\mu}{k} U_3 \quad (16)$$

Introducing the following nondimensional variables as

$$U_i = \frac{\phi_1 u_i}{\rho L} \qquad T = \frac{\rho L^2 t}{\phi_1}$$

$$\phi_2 = \rho L^2 \beta \qquad P = \frac{\phi_1^2 p}{\rho L^2}$$

$$\frac{X_i}{L} = x_i \qquad \frac{Y_i}{L} = y_i$$

$$\phi_3 = \rho L^2 \upsilon_c \qquad A_i = \frac{\phi_1^2 a_i}{\rho^2 L^3}$$

$$S_{i,j} = \frac{\phi_1^2 s_{i,j}}{\rho L^2} \qquad E_{i,j}^{(1)} = \frac{\phi_1 e_{i,j}^{(1)}}{\rho L^2}$$

$$E_{i,j}^{(2)} = \frac{\phi_1^2 e_{i,j}^{(2)}}{\rho^2 L^4} \qquad k = \frac{\rho L^3}{\phi_1^2 K}$$

where $T$ is the (dimensional) time variable, $\rho$ the mass density, and $L$ the characteristic length.

A class of plane flows given by the velocity components has been considered in the directions

$$u_1 = u(y,t) \text{ and } u_2 = 0 \quad (17)$$

of rectangular cartesian coordinates $x$ and $y$. The velocity field given by Eqn (17) identically satisfies the incompressibility condition. The stress can now be obtained in the nondimensional form as

$$s_{xx} = -p + \upsilon_c \left(\frac{\partial u}{\partial y}\right)^2 \quad (18)$$

$$s_{yy} = -p + (\upsilon_c + 2\beta)\left(\frac{\partial u}{\partial y}\right)^2 \quad (19)$$

$$s_{xy} = \frac{\partial u}{\partial y} + \beta \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right) \quad (20)$$

In view of the above, the equations of motion in the present case of porous boundary yield:

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \beta \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial y^2}\right) - \frac{1}{K} u \quad (21)$$

and $$0 = -\frac{\partial p}{\partial y} + (2\beta + \upsilon_c)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 \quad (22)$$

Equation (21) shows that $-{\partial p}/{\partial x}$ must be independent of space variables, and hence, may be taken as $\xi(t)$. Equation (22) now yields:

$$p = p_0(t) - \xi(t)x + (\upsilon_c + 2\beta)\left(\frac{\partial u}{\partial y}\right)^2 \quad (23)$$

Considering $\xi(t) = 0$, the flow characterised by the velocity is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \beta \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial y^2}\right) - \frac{1}{K} u \quad (24)$$

where $K$ is the nondimensional porosity constant. It may be noted that the presence of $\beta$ changes the order of differential from two to three.

## 3. DISTURBANCE OF A LIQUID AT REST DUE TO THE SINUSOIDAL OSCILLATIONS OF THE BOTTOM

The oscillations of a classical viscous liquid on the upper half of the plane $y \geq 0$ with the bottom oscillating with the velocity $\alpha e^{i\sigma t}$ is examined in the present case. The motion of the second-order fluid is governed by Eqn (24 ) with boundary conditions

$$u(0,t)=\alpha e^{i\sigma t} \tag{25}$$

$$u(\infty,t)=0 \tag{26}$$

Assuming the trial solution as

$$u(y,t)=\alpha e^{i\sigma t} f(y) \tag{27}$$

$$f''(y)=p^2 f(y) \tag{28}$$

where

$$p^2=\frac{i\sigma+\frac{1}{K}}{1+i\beta\sigma}=\frac{(\beta\sigma^2+\frac{1}{K})+i(\sigma-\frac{\beta\sigma}{K})}{(1+\beta^2\sigma^2)} \tag{29}$$

When expressed in polar form, one has:

$$p=r\left[(\cos(\frac{\pi}{4}-\frac{\varepsilon}{2})+i\sin(\frac{\pi}{4}-\frac{\varepsilon}{2})\right]$$

$$r=\frac{[(\beta\sigma^2+\frac{1}{K})^2+(\sigma-\frac{\beta\sigma}{K})]^{1/4}}{\sqrt{(1+\beta^2\sigma^2)}}$$

$$\varepsilon=\tan^{-1}(Q) \text{ and } Q=\frac{\frac{1}{K}+\beta\sigma^2}{\sigma-\frac{\beta\sigma}{K}} \tag{30}$$

Also, the conditions satisfied by $f(y)$ are:

$$f(0)=1,\ f(\infty)=0 \tag{31}$$

This yields the solution:

$$f(y)=\exp\left[(-yr\left\{(\cos(\frac{\pi}{4}-\frac{\varepsilon}{2})+i\sin(\frac{\pi}{4}-\frac{\varepsilon}{2})\right\}\right] \tag{32}$$

hence

$$u(y,t)=\alpha\exp\left[(i\sigma t-yr\left\{(\cos(\frac{\pi}{4}-\frac{\varepsilon}{2})+i\sin(\frac{\pi}{4}-\frac{\varepsilon}{2})\right\}\right] \tag{33}$$

The flow is thus represented by standing transverse wave with its amplitude rapidly diminishing with increasing distance from the plane. This phenomenon is independent of $\nu_c$ as noticed for all two-dimensional flows.

The magnification factor $A^*$ of the amplitude of this wave, wrt the amplitude of the disturbance ($\alpha$), may be written as

$$(A^*)^2=[\text{Real part of } u(y,t)]^2+[\text{Imaginary part } u(y,t)]^2 \tag{34}$$

$$A^*=\alpha\exp\left[-y\sqrt{r}\cos(\frac{\pi}{4}-\frac{\varepsilon}{2})\right] \tag{35}$$

which is in the form $A^*=e^{-xy^*}$

$$\text{where}\quad \chi y^*=\frac{y\sqrt{r}}{\sqrt{2}}\left[\cos\frac{\varepsilon}{2}+\sin\frac{\varepsilon}{2}\right] \tag{36}$$

$$\text{where}\quad \chi=\frac{1}{(1+\beta^2\sigma^2)^{\frac{1}{4}}}\sqrt{\frac{Q+\sqrt{1+Q^2}}{1+Q^2}} \tag{37}$$

and

$$y^*=\frac{y(1+\beta^2\sigma^2)^{\frac{1}{4}}}{\sqrt{2}}\left[\frac{(\frac{1}{K}+\beta\sigma^2)^2+(\sigma-\frac{\beta\sigma}{K})^2}{(1+\beta^2\sigma^2)^2}\right]^{\frac{1}{4}} \tag{38}$$

## 4. DISCUSSION AND CONCLUSIONS

When the bounding surface is non-porous, ie, as, $K \to \infty$, $y^* = y\sqrt{\left(\frac{\sigma}{2}\right)}$. It may be noticed that from Fig.1, $\chi = 1$ in the Newtonian case ($\beta$ = 0). Also when $Q = Qc = 1.54$, ie, when $\beta = \frac{Qc}{\sigma}$. Hence for each second-order fluid, there exists a frequency distribution so that the velocity response is the same as that in the Newtonian flow. It can be seen that for each $\chi$, there correspond two values of $Q$ which lie in between 1 and $Qc$ when $\chi > 1$. Also, these two values of $Q$ are of opposite signs when $\chi < 1$.

In the presence of elasticoviscosity parameter ($\beta$) and porosity value ($K$), $y^*$ is as given in Eqn (38). Further, it may be noted that from Fig.2, $\chi = 4$. The interesting part in this situation is $Q = Qc = 1.54$, ie, when

$$\beta = \frac{Qc\sigma - \frac{1}{K}}{\sigma^2 - \frac{Qc\sigma}{K}}$$

Therefore, for each second-order fluid (ie for a given $\beta$ and as $K \to \infty$), there exists a frequency distribution so that the velocity is the same as that in the Newtonian flow. It can be seen that for each $\chi$, there correspond two values of $Q$ which lie between 4 and $Qc$ when $\chi > 4$. Also, these two values of $Q$ are of opposite signs when $\chi < 4$. The nature of the profiles for various values of $Q$ and $\chi$ does not change the qualitative character, except for the magnification.

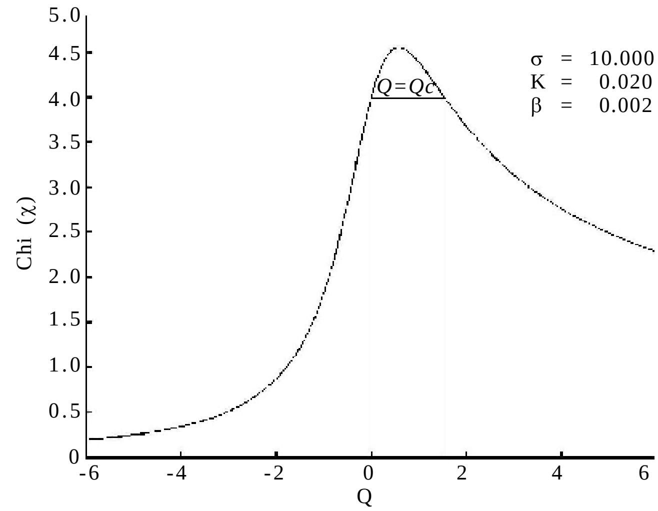


**Figure 2. Variation of $\chi$ with $Q$ considering elasticoviscosity and porosity factors.**

The variation of velocity profiles for various values of the porosity factor are illustrated in Fig.3. It is seen that the velocity decreases as one moves away from the plate. Further, as the porosity factor ($K$) increases, the velocity also increases.

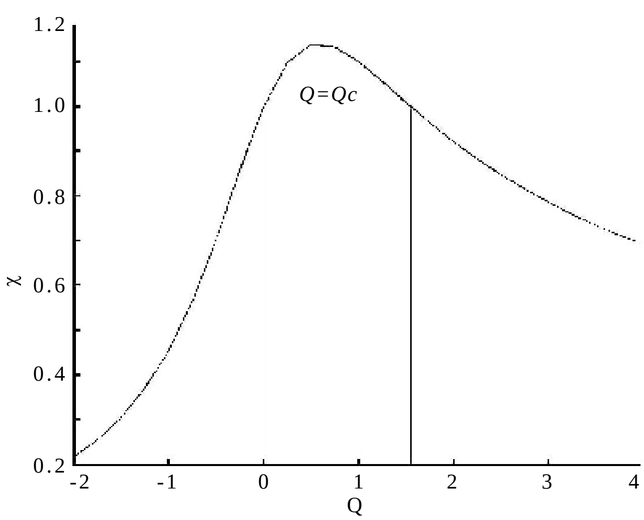


**Figure 1. Variation of $\chi$ with $Q$.**

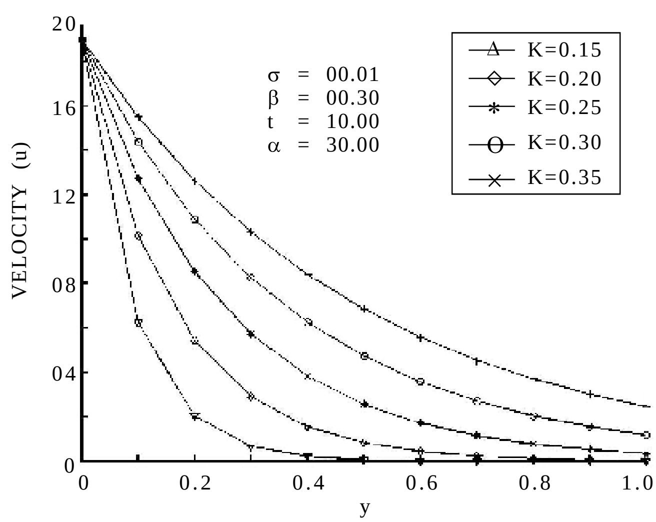


**Figure 3. Effect of porosity on the velocity profiles.**

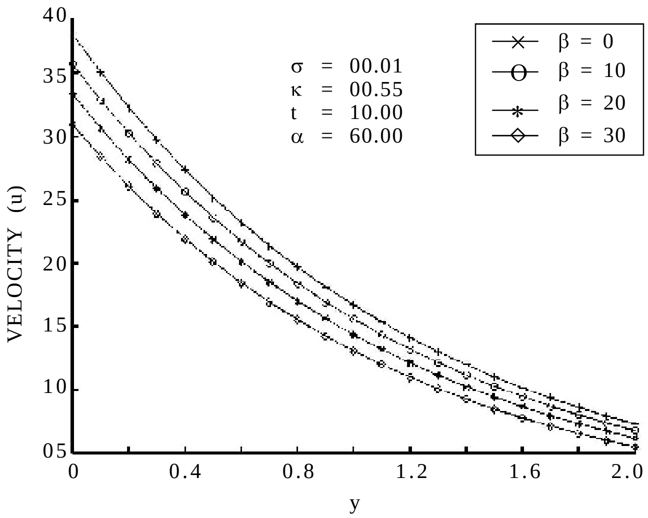


**Figure 4. Effect of elasticoviscosity on the velocity profiles.**

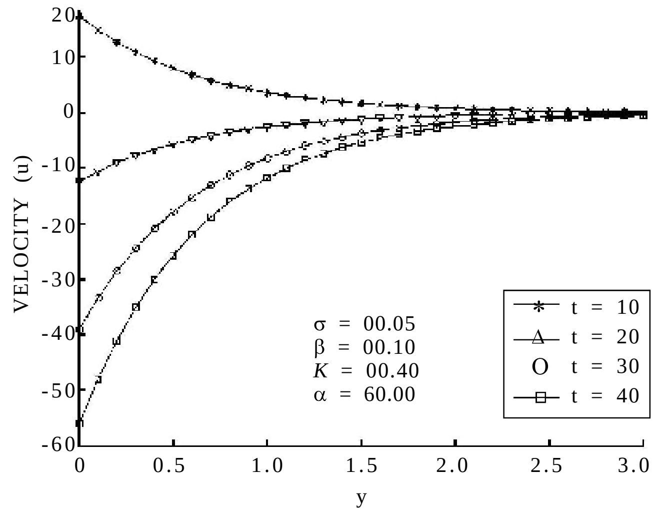


**Figure 6. Effect of time parameter on the velocity profiles.**

Figure 4 illustrates the velocity profiles for different values of elasticoviscosity parameter ($\beta$). It has been observed that as $\beta$ increases, there is a decreasing trend in the velocity profiles. This can be attributed to the strong intramolecular forces.

The nature of velocity profiles for different values of frequency of excitation ($\sigma$) are illustrated in Fig. 5. It has been observed that as the frequency of the excitation is increased, there is a decreasing trend in the velocity at the boundary region. Further, it has also been observed that there is a back flow in the neighbourhood of the plate which subsequently settles down as one moves away from the plate.

The effect of time ($t$) on the nature of velocity profiles is seen in Fig. 6. As $t$ increases, there is a decreasing trend in the velocity profiles near the boundary layer. Further, for certain values of $t$, even back flow is also observed. However, the flow field settles down as one moves away from the plate.

Figure 7 illustrates the effect of elasticoviscosity parameter ($\beta$) on the magnification factor ($A^*$). It is seen that as $\beta$ increases, there is an increase in the magnification factor.

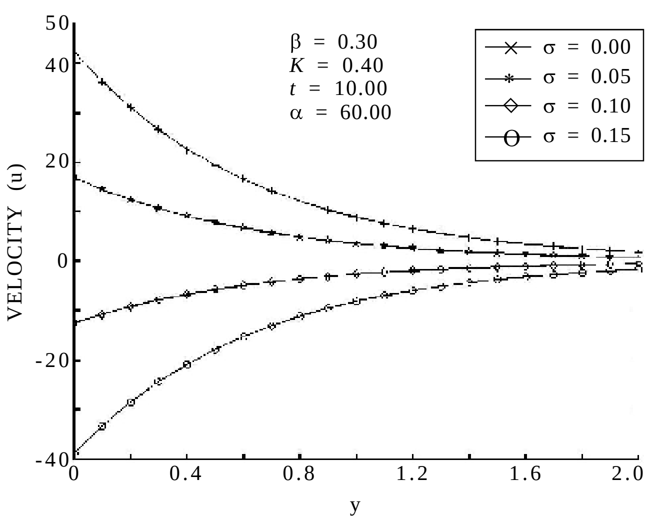


**Figure 5. Effect of frequency of excitation on the velocity. profiles.**

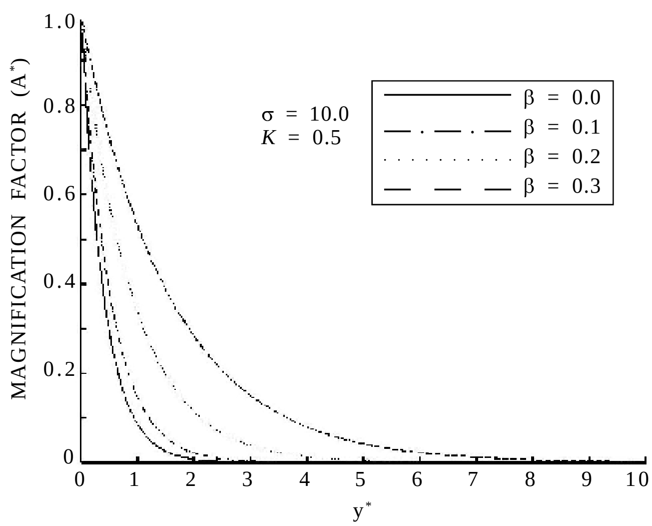


**Figure 7. Effect of elasticoviscosity on the magnification factor.**

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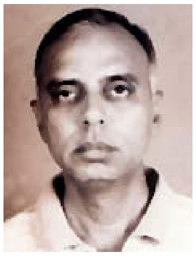
## Contributors



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