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SHORT COMMUNICATION

Normal Approximation to a Sum of Geometric Random Variables with Application to Ammunition Stockpile Planning

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ABSTRACT

The normal approximation for a sum of geometric random variables has been examined. This approximation is relevant to the determination of direct-fire ammunition stockpile levels in a defence setting. Among the methodologies available for this assessment, one is a target-oriented methodology. This approach calculates the number of rounds necessary to destroy a given fraction of the enemy force and infrastructure. The difficulty is that the number of rounds required cannot be determined analytically. An obvious numeric approach is Monte Carlo simulation. Another is the approximation approach which has several advantages like it is easy to implement. and is accurate even in the case where the number of targets is low.

Keywords: Defence planning, defence planners, single-shot kill probability, ammunition stockpiles, direct-fire weapon systems, DF Opstock

1. INTRODUCTION

It is well known that a sum of random variables can be approximated by a normal random variable. The normal approximation for a sum of geometric random variables has been examined. This approximation is relevant for defence planners who must determine ammunition stockpiles for direct-fire weapon systems (DF Opstock). These inventories are necessary in the face of long lead-times to purchase ammunition relative to the short lead-times for conflict.

Among the methodologies available for the assessment of DF Opstock, one is termed the target-oriented methodology[1] (TOM). This approach calculates the number of direct-fire rounds necessary to destroy a given fraction of the enemy force and infrastructure with a fixed high probability. In contrast, the literature has tended to focus on the extent of damage when a fixed number of rounds is fired. A good summary of this literature can be found in Jaiswal[2].

The difficulty in using a TOM approach is that the number of rounds cannot be determined analytically. An obvious numeric approach is Monte Carlo simulation. Still another is the approximation presented here. Our approximation has several advantages. First, it is easy to implement and second, it appears to be accurate even in the case where the number of targets is low.

2. PROBLEM

2.1 Single Target Type

Suppose it is necessary to destroy m targets of a particular type. Following Helmbold[3], it is

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assumed that planners have identified the appropriate breakpoints for a given operation and these breakpoints are consistent with having to destroy *m* targets. The single-shot kill probability (SSKP) for each round is *s* assumed to be constant over all rounds fired. How many rounds, *r*, need to be fired to give a high probability that all *m* targets are killed?

In principle, the solution of the problem is straightforward. Imagine that there are *m* real targets and *r*–*m* imaginary targets for a total of *r* targets. A single round is assigned to each of these *r* targets, rounds are fired, and then the number of kills is counted. The probability of destroying at least *m* of these real and imaginary targets is equivalent to the probability of destroying the *m* real targets and is given by a sum of binomial probabilities as

$$K(r,m) = \sum_{i=m}^{r} \binom{r}{i} s^i (1-s)^{r-i} \tag{1}$$

A minimal inventory level, $r = r^*_\alpha$ is sought which makes $K(r,m)$ at least α, a predetermined high probability. The difficulty is that one cannot solve for this inventory level analytically. In the development herein, it is shown how to get an approximate solution using the normal approximation to a sum of geometric random variables.

Suppose one considers each of the *m* targets in sequence. One will destroy them one at a time until all *m* are destroyed. Define the random variable R_j to be the number of rounds required to kill target *j* where $j = 1,2....,m$.

The total number of rounds required to destroy all *m* targets, then, is

$$R = R_1 + R_2 + ... + R_m \tag{2}$$

Clearly R_J is a geometric random variable. The probability R_J that takes the value *x* is $(1-s)^{x-1}s$. That is, the initial x–1 rounds must miss and the x^{th} must hit. It is well known that this geometric distribution has a mean

$$E(R_j) = \frac{1}{s} \tag{3}$$

and variance

$$Var(R_j) = \frac{1-s}{s^2} \tag{4}$$

If *m* is sufficiently large, *R* is approximated by a normal distribution with mean

$$E(R) = E(R_1 + R_2 + ... + R_m) = m.E(R_j) = \frac{m}{s} \tag{5}$$

and variance

$$Var(R) = Var(R_1 + R_2 + ... + R_m)$$
$$= m.Var(R_i) = \frac{m(1-s)}{s^2} \tag{6}$$

Hence, an approximation of the number of rounds required is

$$\hat{r}_\alpha = \frac{m}{s} + z_\alpha \sqrt{\frac{m(1-s)}{s^2}} \tag{7}$$

where z_α is the ordinate of the standard normal random variable having a cumulative probability α, i.e., if *Z* is normally distributed with mean 0 and variance 1, z_α solves

$$\Pr(Z \le z_\alpha) = \alpha \tag{8}$$

Note that $\hat{r}_\alpha$ requires the addition of two parts: the expected number of rounds, $m/s = E(R)$, and a risk premium,

$$z_\alpha \sqrt{\frac{m(1-s)}{s^2}} \tag{9}$$

The size of this risk premium depends on z_α, *m*, and *s*. It increases with *m* the number of targets that need to be destroyed, and decreases with *s* the accuracy of the weapon system delivering the rounds. One can express the risk premium in percentage terms as follows:

$$\hat{r}_\alpha = \frac{m}{s} + z_\alpha \sqrt{\frac{m^2}{s^2}\frac{(1-s)}{m}} = E(R)\left[1 + z_\alpha \sqrt{\frac{1-s}{m}}\right] \tag{10}$$

Hence the percentage the expected number of rounds must be increased by is

$$z_\alpha \sqrt{\frac{1-s}{m}} \tag{11}$$

To evaluate the accuracy of this approximation, we examined the values of r_α^* and $\hat{r}_\alpha$ for various values of m and s with α = 0.95. Our results are shown in Table 1. In general, the approximation is very good. It is best with high SSKPs and a high number of targets. With most militaries going increasingly to smart direct-fire munitions, munitions for which s is likely to be high, the approximation is almost the exact. Note what happens when the SSKP approaches 1:

$$\lim_{s\to 1} \hat{r}_\alpha = \lim_{s \to 1}\left[\frac{m}{s} + z_\alpha \sqrt{\frac{m(1-s)}{s^2}}\right] = m \tag{12}$$

As one would expect, the number of rounds required approaches m, the number of targets.

Table 1. Exact and approximate number of rounds required to destroy targets sets under various parameter assumptions of SSKP and target set size

# Targets	SSKP	# Rounds r_α^*	$\hat{r}_\alpha$
5	0.1	89	85
	0.3	28	27
	0.5	16	16
	0.7	10	10
	0.9	7	7
10	0.1	155	150
	0.3	49	48
	0.5	28	28
	0.7	19	19
	0.9	13	13
15	0.1	215	211
	0.3	69	69
	0.5	40	40
	0.7	27	27
	0.9	19	19
20	0.1	270	274
	0.3	88	88
	0.5	51	51
	0.7	35	35
	0.9	25	25
100	0.1	1157	1160
	0.3	380	380
	0.5	224	224
	0.7	156	156
	0.9	117	117

2.2 Heterogeneous Target Set

The approximation is easily generalised to multiple target types. Suppose a particular munition must be able to kill m targets of type i, i = 1,2 …., p. The munition's SSKP against targets of type i is s_i. One needs to determine the number of rounds required to kill all of the targets with a given probability. Based on the argument above for a single target type, the number of rounds required, R, is normally distributed with mean

$$E(R) = \frac{m_1}{s_1} + \frac{m_2}{s_2} + \ldots + \frac{m_p}{s_p} = \sum_i \frac{m_i}{s_i} \tag{13}$$

and variance

$$Var(R) = \frac{m_1(1-s_1)}{s_1^2} + \frac{m_2(1-s_2)}{s_2^2} + \ldots + \frac{m_p(1-s_p)}{s_p^2} = \sum_i \frac{m_i(1-s_i)}{s_i^2} \tag{14}$$

Hence to kill all targets with probability α, the required number of rounds can be approximated with

$$\hat{r}_\alpha = \sum_i \frac{m_i}{s_i} + z_\alpha \sqrt{\sum_i \frac{m_i(1-s_i)}{s_i^2}} \tag{15}$$

Here is an example. Suppose one has the following target set with associated SSKPs:

	Target 1	Target 2	Target 3
Number	40	60	100
SSKP	0.4	0.6	0.25

The expected number of rounds required is

$$E(R) = \frac{40}{0.4} + \frac{60}{0.6} + \frac{100}{0.25} = 600 \tag{16}$$

and its variance is

$$\begin{aligned} Var(R) &= \frac{40(1-0.4)}{0.4^2} + \frac{60(1-0.6)}{0.6^2} + \\ &+ \frac{40(1-0.25)}{0.25^2} = 1416.7 \end{aligned} \tag{17}$$

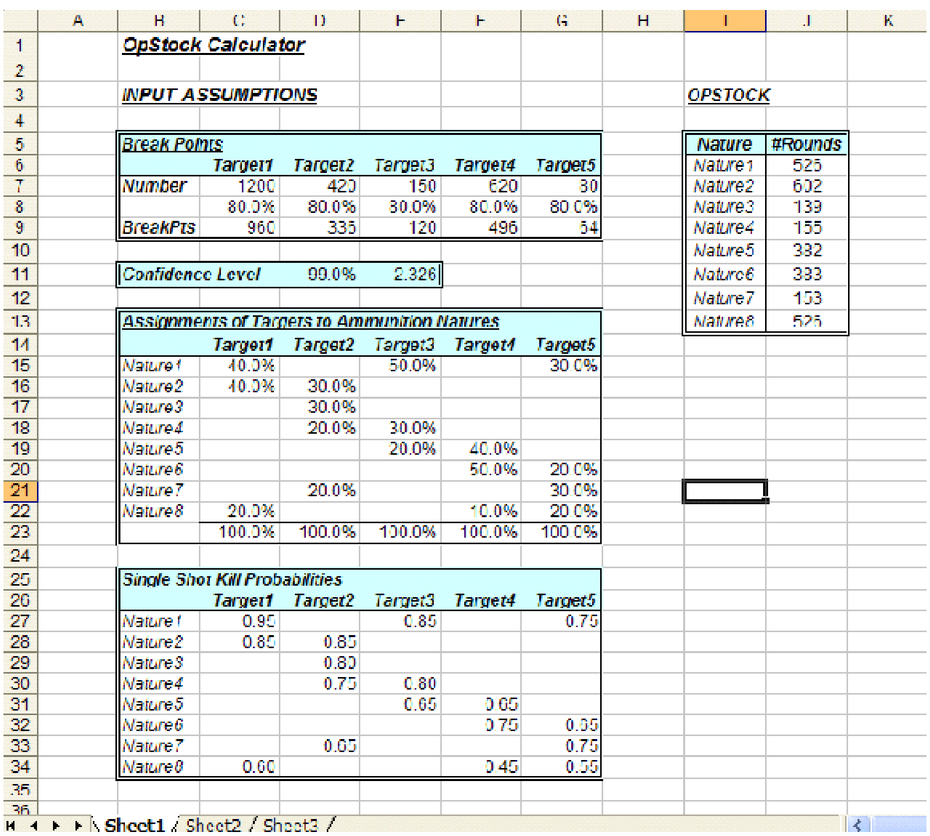
Therefore, an approximation of the number of rounds required to be 95 per cent sure of killing all targets is

$$\hat{r}_{\alpha} = 600 + 1.645\sqrt{1416.7} = 662 \text{ rounds} \tag{18}$$

3. SPREADSHEET IMPLEMENTATION

The approximation and its ease of use enable the calculation of DF Opstock levels within a spreadsheet environment. By way of example, consider the spreadsheet snapshot in Fig. 1. The input assumptions of the model are contained in the four tables on the left. They include:

(a) *Breakpoints*: In this example, there are 5 targets (tanks, APCs, etc.) labeled Target1, Target2, …, Target5. The numbers of each of these are specified in cells C7..G7. Note that there are 1200 of Target1, 420 of Target2, etc. The percentages in the row underneath are the percentages of each target that must be neutralised or destroyed. When these percentages are applied to the target force numbers, one ends up with breakpoints, the planner's estimate of the opposing force which the DF Opstock must be able to neutralise or destroy. These breakpoints are shown in cells C9..G9.



OpStock Calculator

INPUT ASSUMPTIONS

Break Points	Target1	Target2	Target3	Target4	Target5
Number	1200	420	150	620	30
	80.0%	80.0%	80.0%	80.0%	80.0%
BreakPts	960	335	120	496	54

Confidence Level	99.0%	2.326

Assignments of Targets to Ammunition Natures	Target1	Target2	Target3	Target4	Target5
Nature1	40.0%		50.0%		30.0%
Nature2	40.0%	30.0%			
Nature3		30.0%			
Nature4		20.0%	30.0%		
Nature5			20.0%	40.0%	
Nature6				50.0%	20.0%
Nature7		20.0%			30.0%
Nature8	20.0%			10.0%	20.0%
	100.0%	100.0%	100.0%	100.0%	100.0%

Single Shot Kill Probabilities	Target1	Target2	Target3	Target4	Target5
Nature1	0.95		0.85		0.75
Nature2	0.85	0.85			
Nature3		0.80			
Nature4		0.75	0.80		
Nature5			0.65	0.65	
Nature6				0.75	0.55
Nature7		0.65			0.75
Nature8	0.60			0.45	0.55

OPSTOCK

Nature	#Rounds
Nature1	525
Nature2	602
Nature3	139
Nature4	155
Nature5	332
Nature6	333
Nature7	153
Nature8	525

Figure 1. Example of a spreadsheet implementation of the DF Opstock calculation by the target oriented methodology using the normal approximation.

(b) *Confidence Level*: The planner must specify the probability that the calculated DF Opstock will destroy the breakpoint number of targets for each target type. In this example, it is chosen to be 99 per cent (cell D11).

(c) *Assignments of Targets to Ammunition Natures*: The breakpoint population of each target type must be assigned to particular natures. In this example, note that 40 per cent of Target1 is assigned to Nature1, 40 per cent to Nature2, and 20 per cent to Nature8. It is important to note that this assignment, in part, is dictated by the vagaries of conflict. That is, even though one would like to neutralise certain enemy assets with our best systems against those assets, this is not always possible.

(d) *Kill Probabilities*: These are the planner's assumptions about each nature's average probability of neutralising or destroying a particular target type on a single shot.

With these assumptions, it is straightforward to calculate the DF Opstock requirement using the approximations developed above. The DF Opstock is shown on the right-hand side of the spreadsheet snapshot in cells I5...J12.

4. SUMMARY

The purpose of this study is to show that the normal approximation for a sum of geometric random variables results in a very good approximation of DF Opstock using a TOM. Clearly the calculations presented does not include some other important factors, such things as logistic loss and suppression fire rates. However, the purpose has been to show that, at least for simple TOM models, the normal approximation can give an accurate approximation of DF Opstock. Future research will examine the performance of the approximation within richer DF Opstock models.

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