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Analysis of the Brownian Motion Approach for Ballistic Resistance Evaluation Using the Maximum Likelihood Inference

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ABSTRACT

Armor technologists' improvement of protection systems led to the design of complex systems. Given the risk factor on human life, increasing requirements on the ballistic resistance evaluation are imposed. Consequently, an increased effort is dedicated to estimating the perforation probability curve as a function of the bullet impact velocity. The main limitation of methods that fits a normal law to perforation velocities is their purely statistical character. A Brownian-based approach that couples the system response variability and physics was proposed using the Chi-square and Kolmogorov-Smirnov criterion function for model parameters estimation. One major limitation of this inference approach is the large experimental database required for its execution. The contribution of this paper is the introduction of the maximum likelihood inference for parameters estimation of the Brownian-based approach. The agreement between the obtained results and the experimental ones confirms the appropriateness of the likelihood inference to solve the studied problem. Moreover, the estimations uncertainty was analyzed and compared to the existing method ones. It was observed that the proposed model reduces the confidence intervals on key velocity estimations. Accordingly, the present work encourages the adoption of this proposed methodology in a laboratory context with a restrained sample size.

Keyword: Ballistic resistance; Ballistic limit; Likelihood inference; Perforation probability; Uncertainty quantification

NOMENCLATURE

V_i	: Velocity of the bullet at the time step i
V_x	: Impact velocity of the bullet with x % probability of perforation
V_{50}	: Impact velocity of the bullet with 50 % probability of perforation
U	: Binary outcome of the target response, 0 if perforation and 1 if not
V(t)	: Bullet instantaneous velocity
a(V,t)	: Bullet instantaneous deceleration
σ(V,t)	: SDE's diffusion
W(t)	: Wiener or Brownian motion process
X(t)	: Bullet instantaneous position
Z_x	: x percentile of the standard normal distribution
ti	: Time step i
L(a,σ)	: Likelihood function
P	: Probability of perforation
j	: j^{th} sample of the experimental database
H	: Target thickness
R_t	: Effective resisting stress to the penetration
ρ_p	: Impactor density
L_{eff}	: Impactor effective length
χ^2_1	: Chi square law with 1 degree of freedom
$CI_{95\%}$	: Confidence interval with a 5% confidence level, the super and subscript indicate the upper and lower limits of the interval.

1. INTRODUCTION

Ballistic materials are protection materials designed to defeat a variety of ballistic threats such as bullets and fragments. Performance enhancement of these materials has gained considerable attention in recent years. Thus, performance assessment metrics are required during either the materials research and development process or final control quality (acceptance tests and end-of-life tests). The typical key performance indicator is the limit velocity V_0, the maximum bullet velocity at which the considered bullet never perforates the assessed target. Given the complexity of the determination of this maximum impact velocity that still gives zero probability of perforation[1], this measurement is generally reserved for final control quality. Therefore, the V_{50} velocity, where V_{50} is the bullet velocity with 50 % probability of complete penetration of the given target, is used during the earliest stages of material improvement. Several methods have been developed for the estimation of the V_{50} and even the whole curve of the perforation probability as a function of the bullet velocity.

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An exhaustive literature survey is available [2]. These methods analyse the coupled observations of the bullet impact velocity and the target response coded in a binary outcome (occurrence or not of perforation). The STANAG 2920[3] approach gives an estimate of the V_{50} velocity based on small sample size but it does not provide any information about the dispersion of the impact response distribution. Langlie[4], Kneubuehl[5], and Probit[6] implement advanced statistical tools leading to a better description of the system variability. Therefore, in addition to the V_{50} velocity, an estimate of the sample standard deviation σ is provided, under the assumption of the normality of the impact response distribution. Furthermore, the stated assumption allows the estimation of any percentile of interest V_x as follows:

$$V_x = V_{50} - Z_x\sigma \tag{1}$$

where, V_x is the impact velocity corresponding to x % of the probability of perforation and Z_x is the equivalent percentile of the standard normal distribution.

A literature survey reveals that authors[7–9] have extensively discussed these methods' performance. The common conclusion is that existing methods produce significant differences in estimating perforation probabilities at high and low impact velocities (extreme values probabilities) despite the similarities shown when estimating the V_{50}. For the sensitivity analysis of binary outcome experiments, Dixon[10] explained that the normality assumption is only valid around the mean value according to the central limit theorem. This result encouraged investigations for a new modelling paradigm that is not based on the normality of perforation velocities.

Recently, Coghe[11] & Tahenti[12] introduced the concept of stochastic differential equations to model the stochastic response of ballistic impact phenomena. This first implementation used the Chi-square and Kolmogorov-Smirnov goodness of fit tests. The proposed inference requires an experimental estimation of the perforation probability per impact velocity. Therefore, the collected experimental database has to be sufficiently large. Accordingly, the aim of this paper is the development of a maximum likelihood inference tool that extends the application of this Brownian-based approach to samples with a limited size. In parallel, uncertainty quantification on model estimations is executed based on the maximum likelihood inference results under sample size restriction. First, the experimental database is introduced. Then, the stochastic modelling concept is explained. Afterwards, the results for the Brownian-based approach using maximum likelihood inference are presented and discussed.

2. METHODOLOGY

The goal of this stochastic approach is to model the impact response variability based on the bullet motion within the target. Newton's second law is applied to model the penetration process with additional fluctuations issuing from complementary "random" forces. Consequently, a stochastic differential equation (SDE) model is introduced to describe the observed randomness. In the first section, the experimental database is introduced. Next, the proposed modelling concept is explained. Finally, the likelihood inference method is introduced for model parameters determination.

2.1 Experimental Results

The experimental database is composed of ballistic impacts of a 5.56x45 mm NATO projectile against two mild steel plates spaced by a 20 mm gap. Table 1 provides a detailed description of the impact configuration. The first column illustrates the parameters of the used bullet. The second column supplies the parameters of the target.

The experimental observation is the bullet impact velocity and the material response coded in a binary outcome (U=0 if complete penetration takes place and 1 if partial penetration occurs). The number of observations that were recorded is N=581. In addition, the experiments were conducted to identify a zone of mixed results to allow the comparison of the model results with those provided by existing methods. The ballistic resistance of the studied system was characterized using V_1, V_{50} and V_{99} estimations. The subscripts 1, 50 and 99 indicate the equivalent probability of perforation at this impact velocity. Different methods exist in the literature for the V_{50} velocity estimation or even the entire curve estimation of the perforation probability. Table 2 summarizes the estimation results of the established database using the empirical histogram and the Probit[6] method.

Table 1. Configuration parameter

Bullet		Target	
Designation	FN SS109	Front plate thickness	16 mm
Materials	lead core and a steel penetrator in its tip.	Air gap	20 mm
Masse	4,011 ± 0,1 g	Back plate thickness	4 mm
Impact velocity	Measured at 2.5 m before the target	Test plate dimensions	500 mm x 500 mm
Impact condition	Normal to the plate	Material	Mild steel

Table 2. Impact results of two spaced mild steel plates impacted by a 5.56x45 mm NATO Ball.

Velocity method	V_1 (ms^{-1})	V_{50} (ms^{-1})	V_{99} (ms^{-1})
Histogram	488.15	512.47	543.78
Probit	484.92	513.92	542.91

2.2 Brownian-Based Approach

The modelling of the stochastic system behaviour under impact loading is based on the implementation of Newton's second law. The bullet deceleration is the sum of a deterministic term and a fluctuating term. The latter simulates the random resistance to the bullet penetration to reproduce all its possible paths within the target. Accordingly, the bullet motion within the target is governed by the following stochastic differential Eqn. (SDE):

$$dV(t) = a(V,t)dt + \sigma(V,t)dW(t) \tag{2}$$

where, $a(V,t)$ is the drift coefficient which defines the bullet deceleration, $\sigma(V,t)$ is the SDE's diffusion that will reproduce the system response variability and $W(t)$ is the Wiener process. Accordingly, the bullet motion is described by the following system:

$$\begin{cases} dV(t) = a(V,t)dt + \sigma(V,t)\xi(t)dt \\ X(t) = V(t)dt \\ V(0) = V_i,\ X(0) = 0 \end{cases} \tag{3}$$

where, $X(t)$ is the instantaneous bullet position within the target material. The initial state is defined by the bullet position X=0 and the impact velocity $V=Vi$ at the front face of the target. The diffusion process $V(t)$ solution of the SDE defined in Eqn. 2 is :

$$V(t) = V(0) + \underbrace{\int_0^t a(V,s)ds}_{\text{I}} + \underbrace{\int_0^t \sigma(V,s)dW(s)}_{\text{II}} \tag{4}$$

The term I in Eqn. 4 is a deterministic integral of a bounded function (the bullet deceleration). The Riemann–Stieltjes approximation leads under a given time step discretization of the interval [0,t] to :

$$I = \sum_{i=1}^{n} a(t_{i-1}, V_{i-1})(t_i - t_{i-1}) = \sum_{i=1}^{n} a(t_{i-1}, V_{i-1})\Delta t \tag{5}$$

where, $t_i \in [0,t], i = \{0,1,\ldots,n-1, n = t/\Delta t\}$. The integral II is a stochastic integral. Itô stochastic integration of the Wiener process is equivalent to the Riemann-Stieltjes sum where the integrand is evaluated at the left endpoint of the observation interval $[t_{i-1}, t_i]$:

$$II = \sum_{i=1}^{n} \sigma(t_{i-1}, V_{i-1})(W_i - W_{i-1}) = \sum_{i=1}^{n} \sigma(t_{i-1}, V_{i-1})\Delta W_{t_{i-1}}^{t_i} \tag{6}$$

Applying the previous calculation on a time step interval, $[t_{i-1}, t_i]$, leads to the Euler-Maruyama scheme for numerical integration. Then, Eqn. 4 is discretized as:

$$\begin{cases} V(t_i) = V(t_{i-1}) + a\Delta\Delta + \sigma\Delta W_{t_{i-1}}^{t_i} \\ X(t_i) = X(t_{i-1}) + V(t_{i-1})\Delta t + \frac{a}{2}\Delta t^2 + \sigma\Delta W_{t_{i-1}}^{t_i}\Delta t \end{cases} \tag{7}$$

where, the bullet deceleration $a(t_{i-1}, V_{i-1})$ is considered constant according to the Robins-Euler formula. This choice is motivated by the research findings of Rosenberg[13]. The diffusion coefficient is assumed constant to simplify this first implementation of stochastic processes to model the problem under consideration. A random bullet trajectory is provided for each simulated Brownian path. For this bullet/target combination, the impact duration is of the order of magnitude of 1 ms as a maximum limit. Thus, a time step $dt=T/2^9$=0.2 ms^{-1} is utilized in this numerical integration which provides the same integration parameters[12]. Physically, the bullet's instantaneous velocity is limited to the interval [0, Vi]. For this reason, the bullet motion computation is stopped at the first-time step that yields a negative velocity which prevents the computation of non-physical data (which is known as the stopping time that is equivalent to the drop of the bullet velocity to zero). Furthermore, the comparison of the depth of penetration X with the total target thickness is used to identify the occurrence or not of target perforation.

2.3 Maximum Likelihood Inference

The functional form of the SDE system that describes the bullet motion is supplied. However, the parameters (a) and (σ) are still unknown. The inverse problem is addressed. In the first step, Tahenti[12] applied the Chi-square and Kolmogorov-Smirnov goodness-of-fit tests to find estimators of (a) and (σ). The disadvantage of this approach lies in the need to have experimental estimations of the perforation probability at a given set of bullet impact velocities. Thus, large sample size is required for sufficiently precise estimations. In the present work, the maximum likelihood inference is proposed to detect the model parameters that better reproduce the observed stochastic behaviour of the impact events. Langlie[4] already applied the likelihood inference on one-shot item tests involving N experimental observations as follows:

$$L(a,\sigma) = \prod_{j=1}^{N} P(U_j | V_{ij}) \tag{8}$$

where, the conditional probability of perforation/ no-perforation, $P(U_j \backslash V_{ij})$, is defined like:

$$P(U_j | V_{ij}) = P_j = U_j P(U_j = 1 | V_{ij}) + (1 - U_j) P(U_j = 0 | V_{ij}) \tag{9}$$

The computation of the likelihood function requires the evaluation of the perforation probability, P_j, for all the observation couples (V_{ij}, U_j) of the sample. Langlie [3] used the normality assumption of the perforation probability to compute P_j where the unknown distribution parameters are determined by the likelihood inference. The idea in this implementation is to use the stochastic model for P_j estimation. Indeed, for each observation couple (V_{ij}, U_j), the conditional probability $P(U_j \backslash V_{ij})$ is computed numerically based on the stochastic model using Monte Carlo simulations. The higher the number of Monte Carlo replications N_{mc} is, the more accurate the numerical estimation of the probability P_j is. In this work, $N_{mc}=10^5$ Brownian paths are generated. The perforation probability is computed by dividing the number of detected perforations by the total number of observations N_{mc}.

The parameter (a, σ) selection is governed by the maximization of the probability $L(a, \sigma)$ that the stochastic model generates the experimentally observed sample (V_{ij}, U_j). Therefore, the maximization problem of $L(a, \sigma)$ associated with the variables (a, σ) has to be solved. The advantage of the likelihood inference lies in its direct application to physical measurements rather than using results of observations post-processing as already advanced in Tahenti[12].

3. RESULTS AND DISCUSSION

In the subsequent section, the model results based on the maximum likelihood inference technique will be displayed and discussed. In the first step, the model behaviour regarding the used sample size is examined by applying the inference technique to the complete sample and a reduced subset of the initial sample. Later, the stochastic model results are compared to Probit estimations. Obtaining a good agreement between the stochastic model and existing models is evidence of the appropriateness of the implemented likelihood inference technique for this modelling approach. Additionally, confidence intervals on the model's parameters estimation are established to inspect their effect on key velocities estimations. Again, the comparison of the obtained results with Probit results permits the evaluation of the model performance using the maximum likelihood inference technique.

3.1 Inference Results

One major constraint in ballistic performance assessment is the limited number of test items. For this reason, the statistical inference is tested both on the complete available database and a reduced subset of it. It is useful to be mentioned that the reduced sample contains N=20 ballistic impacts (9 perforations and 11 non-perforations shots). Table 3 summarizes the statistical inference results. The different outcomes of the statistical estimation (the values of â and $\hat{\sigma}$) are displayed in Table 3. In addition, for each parameter estimation, the complete and reduced sample inference is referenced, respectively, under the full and subset label in the first column of Table 3. Finally, the maximum likelihood estimator (MLE) is the location of the best-found maximum of the likelihood function (MLF).

Table 3. Inference results for the stochastic model's parameters

Database	Parameter	MLE	MLF
Full	â	2.3610×10^7	1.0733×10^{-97}
	$\hat{\sigma}$	4567.05	
Subset	â	2.3718×10^7	9.2609×10^{-5}
	$\hat{\sigma}$	4250	

To verify the model results, the research findings of Rosenberg[13] are used. In short, Rosenberg established an analytical formula for the estimation of the V_{50} velocity based on numerical simulation results of impact phenomena. The following Eqn. was obtained:

$$V_{BL} = V_{50} = \sqrt{\frac{2HR_t}{\rho_p L_{eff}}} \qquad (10)$$

where, R_t is the effective resisting stress, ρ_p is the impactor density and L_{eff} is the impactor effective length. Furthermore, in[14] it was proven that this formula is still valid for the impact of metallic plates by rigid projectiles, as is the case in this work. In fact, regarding the overall process, the constant effective resisting force R_t delivers the same global work to the system, despite that in reality the bullet deceleration during the penetration process is time-dependent. Hence, the average of the bullet deceleration can be estimated using:

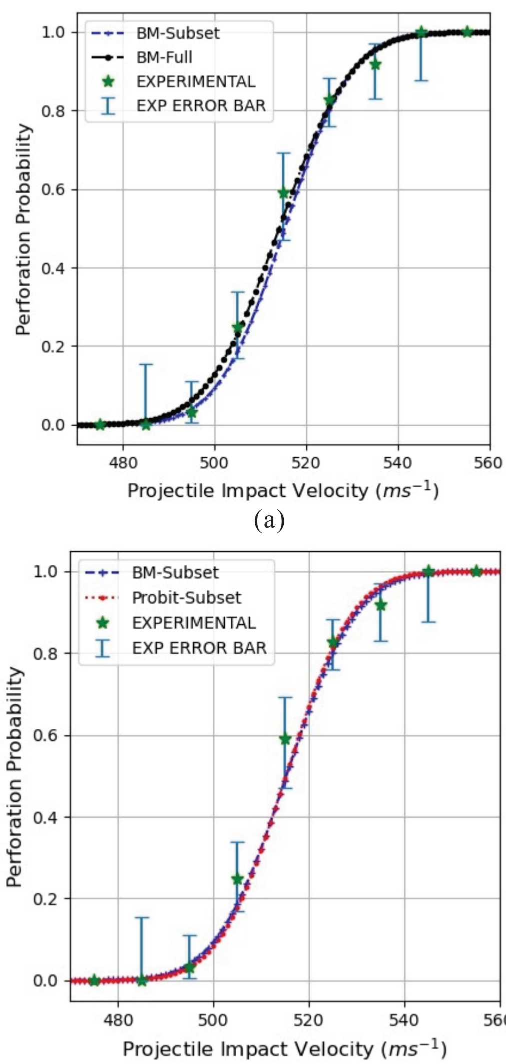
$$a = \frac{R_t}{\rho_p L_{eff}} = \frac{V_{50}^2}{2H} \qquad (11)$$

Considering the experimentally estimated ballistic limit $V_{50} \approx 513 ms^{-1}$ and the target thickness H=5.6 mm, the average deceleration is computed using Eqn. 11 and is equal to a=2.35x10^7 ms^{-2}. The maximum relative error between the model results and the average deceleration is thus 0.9 % (regarding the results of the subset and the complete databases). We conclude that the results of the maximum likelihood inference for the stochastic model are in good agreement with Rosenberg's calculations. Then, the obtained results based on the maximum likelihood inference are in line with the theoretical predictions of the deceleration parameters. Next, the analysis of the stochastic model performance will be based on the comparison of the perforation probability estimations with the experimental and existing methods ones.

3.2 Perforation Probability Estimations

The maximum likelihood inference is fundamental for parameters estimation of the Brownian motion approach using an experimental database with a limited size. The bullet deceleration estimation was verified using the Rosenberg result. A second verification tool of this inference method may be based on the comparison of the perforation probability estimation with experimental and Probit results. For each impact velocity, the numerical resolution of the equations system 7 outputs a simulation of the bullet motion within the target. Several replications of this simulation deliver an estimation of the perforation probability. Accordingly, the mean of the perforation probability is evaluated for each impact velocity based on Monte Carlo simulations. First, this procedure is executed using the parameters estimation of the Brownian motion approach with the full experimental database. Next, the same procedure is applied to a reduced subset of the experimental database. In parallel, the Probit method is applied to this same reduced subset for comparison purposes.

Figure 1 displays the perforation probability curves as a function of the impact velocity. Figure 1(a) serves for the analysis of the database size effect on the model parameters estimation using the maximum likelihood inference. It displays the obtained results using the stochastic model applied to the full experimental database and its subset. The black (dash-dot) and the blue (dashed) curves represent the obtained results using the full and the subset databases, respectively. Alongside, the point estimations of the experimental ones per class of impact velocities and their corresponding 95 % confidence



(a)
(b)

Figure 1. Probability of perforation versus the projectile impact velocity.

intervals are displayed. The Clopper and Pearson method for binomial distributions[15] is called for the construction of the confidence intervals. It can be noted that the two curves pass by the confidence intervals of the experimental estimations of the perforation probability. Moreover, the full database implementation detects more variability in the target response. Effectively, the transition of the relative probability curve from 0 to 1 requires a slightly wider interval of impact velocities. Figure 1(b) represents the perforation probability estimation as a function of impact velocities based on the stochastic (the dashed/blue curve) and the Probit (the dotted/red curve) methods using a subset of the experimental database.

The excellent agreement between the results testifies the competing performance of the stochastic model with the maximum likelihood inference. The main limitation of the Probit method is that it is purely statistical. Conversely, this stochastic model implements an analytical formulation of the physical phenomena with a stochastic term to describe the observed randomness. This methodology can then follow the progress in the field of penetration mechanics by incorporating new knowledge on bullet deceleration, the randomness of the phenomenon regarding the target thickness, the bullet velocity and the availability of velocity measurements during the penetration process.

3.3 Uncertainty Quantification

The next move in the model analysis will be the quantification of the model uncertainty. Point estimates are meaningless without information about their corresponding CIs. Maldague[7], already, computed the confidence intervals (CI) relative to key velocities estimations (V_1, V_{50}, V_{99}) using the Probit method. He pointed out that the main drawback of methods based on the normality assumption of perforation velocities is the large CIs on the estimations using small experimental samples. The same analysis is conducted hereafter to examine if the proposed methodology reduces the CIs on these estimates or not. For this reason, profile likelihood and likelihood ratio concepts are introduced to be applied in the present context.

3.3.1 Uncertainty on Model Parameters Estimations

For the implementation of the Brownian motion model using the small sample case, Fig. 2 shows the likelihood function plot while Fig. 3 shows the negative log-likelihood

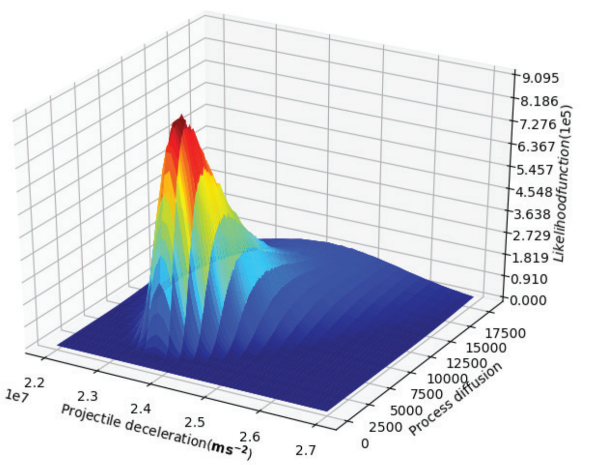


Figure 2. Likelihood function plot.

function contour plot as a function of the bullet deceleration and diffusion coefficient of the penetration process.

The outputted plots are the outcome of a grid search optimization applied to the following design space: $a \in [2.2\times10^7, 2.7\times10^7]$ and $\sigma \in$ [0,18*10^3]. On the likelihood curve, Fig. 2, higher points are estimates for model parameters that best fit[16] the experimental results. The curve is steep and relatively symmetric regarding the bullet deceleration coefficient. So, as the parameter estimation moves from the maximum value, the fitting quality degrades rapidly. On the other side, the likelihood curve is flat and skewed regarding the diffusion coefficient dependence. Then, it is harder to find the diffusion parameter that best fits the experimental data. Given that confidence intervals denote the parameter ranges for which the model still fits the data with a given tolerance limit, the confidence interval on the process diffusion is expected to be larger than the one related to the bullet deceleration. To establish the CIs on the model parameters using likelihood inference, the profile

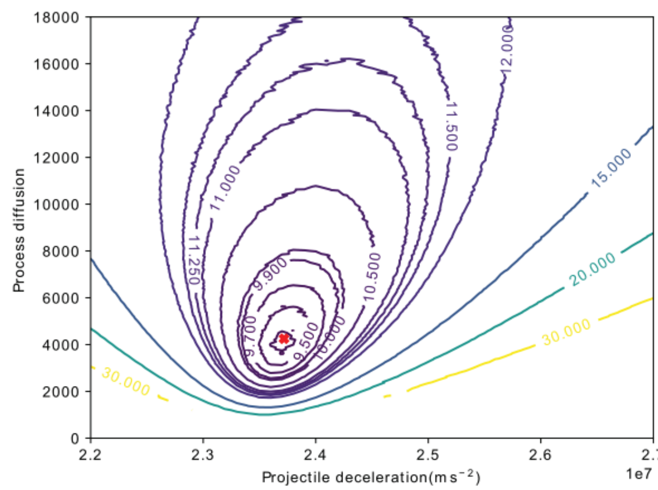


Figure 3. Contour plot of the negative log-likelihood function.

likelihood and the likelihood ratio concepts are called.

In the case of small sample use with no guarantee regarding the normality of the maximum likelihood estimator, it is more accurate to use the likelihood ratio test for CIs construction[17]. Indeed, to determine the CI on the process drift, a, the likelihood ratio states that:

$$2\left(\log\left(L(a,\sigma)\right) - \log\left(L_\sigma(a)\right)\right) \backsim \chi_1^2 \quad (12)$$

the difference $\log(L(a,\sigma))-\log(L_\sigma(a))$ multiplied by two follows a χ_1^2 law with 1 degree of freedom where, $L(a,\sigma)$ is the global minimum of the negative log-likelihood and $L_\sigma(a)$ is the likelihood profile related to the deceleration parameter, a. This later is formed by fixing a to a given range of values and minimizing the negative log-likelihood in the function of σ for each a. Going back to Fig. 3, The contours mark the evolution of the negative log-likelihood function regarding the parameters (a, σ) in the design space. Accordingly, the profile likelihood for the process parameters a (or σ) is established where for each value of a (or σ) the minimum value of the negative log-likelihood is sought along the vertical (or horizontal) line. Figure 4(a) displays the likelihood profile dependence on the bullet deceleration, a, while Fig. 4(b) presents the negative log-likelihood function evolution over the diffusion σ. We

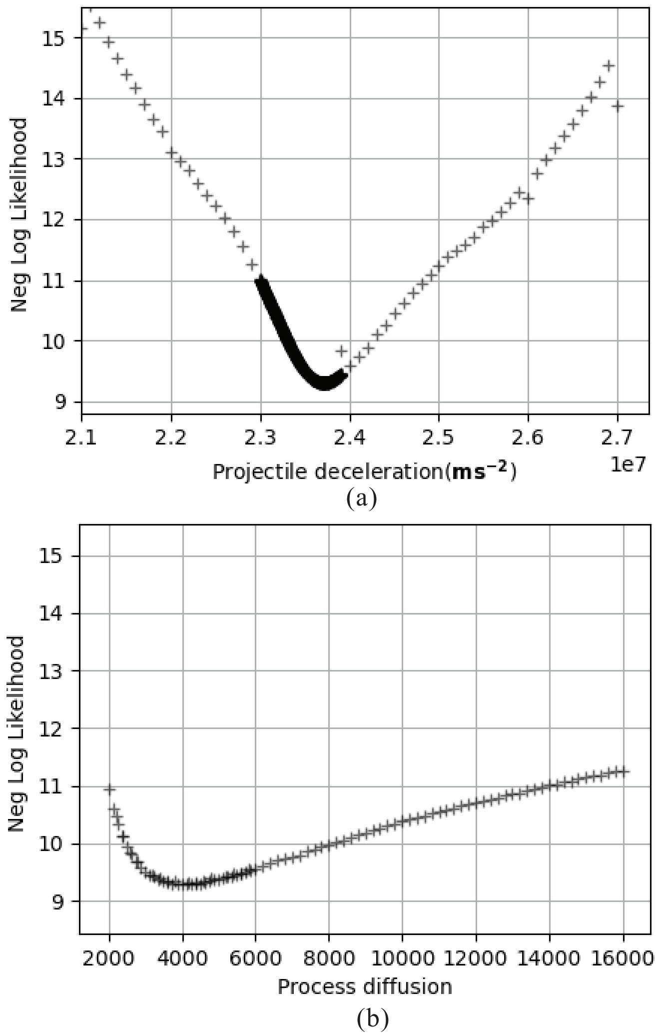


Figure 4. Likelihood profile related to the model parameters.

would like to mention that the χ_1^2 law degree of freedom is computed by making the difference between the total number of the model parameters (equal to two) and the number of fixed parameters for each profile likelihood (equal to one).

The likelihood ratio test specifies the profile likelihood limits on the drift parameter, *a*, that satisfy the little differences requirements regarding the global unrestricted obtained minimum of the NLL (log(*L*(*a*,σ))). This is equivalent to Reference[16]:

$$-\log\left(L_\sigma(a)\right) < \frac{\chi^2_{1-\alpha,1}}{2} - \log\left(L(a,\sigma)\right) \tag{13}$$

where, α is the confidence level chosen equal to 5 % in this case with a related critical value of the Chi-square variable equal to $\chi^2_{0.95,1}$=3.84. Thus, the cut-off of the likelihood profiles is equal to $\frac{\chi^2_{0.95,1}}{2} - \log\left(L(a,\sigma)\right) = 11.21$. Curves in Fig. 4 learn that the confidence intervals are [2.29x10^7, 2.5x10^7]ms^{-2} and [1928, 15836] on the drift *a* and diffusion σ parameters, respectively.

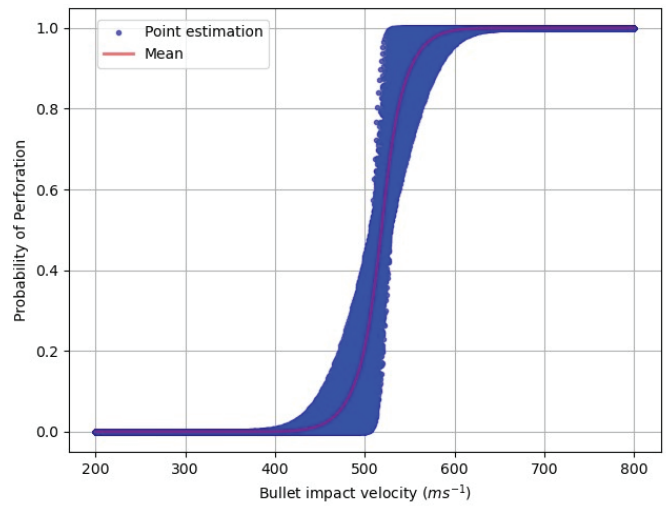


Figure 5. Variability of the perforation probability with the bullet impact velocity.

Now, we move to quantify the effect of these parameter estimation uncertainties on key velocities estimations V_1, V_{50}, and V_{99}.

3.3.2. *Confidence Intervals on Key Velocities Estimations*

Now, the confidence intervals on key velocities estimation, under sample size restriction, may be established. The goal is to verify if the proposed methodology improves the uncertainty on key velocities estimation regarding existing methods estimations. Again, we propagate the model parameters' uncertainty to the outputted results using Monte Carlo error propagation. The model drift and diffusion are uniformly selected from their corresponding confidence intervals. Then, for each couple (a, σ), the perforation probability curve is estimated.

Figure 5 shows all the possible point estimations of the perforation probability for each impact velocity, $V_i \in [200,800] ms^{-1}$. Alongside, the red curve illustrates the mean perforation probability for each impact velocity. Maldague[7], computed the confidence intervals on key velocities estimation (V_1, V_{50}, and V_{99}) while inspecting the normality assumption for

Table 4. Comparison of V_1, V_{50}, and V_{99} confidence intervals estimation using the stochastic models and the Probit methods under sample size restriction

Velocity method		V_1 (ms^{-1})		V_{50} (ms^{-1})		V_{99} (ms^{-1})	
		$CI^{Low}_{95\%}$	$CI^{High}_{95\%}$	$CI^{Low}_{95\%}$	$CI^{High}_{95\%}$	$CI^{Low}_{95\%}$	$CI^{High}_{95\%}$
Histogram		488.150		512.472		543.783	
Probit	Value	400	580	507.88	522.52	450	630
	Deviation	18,07 %	18,82 %	0,9 %	1,96 %	17,25 %	15,86 %
Brownian approach	Value	404	508	505	528	524	623
	Deviation	17,24 %	4,07 %	1,46 %	3,03 %	3,64 %	14,57 %

perforation probability estimation under sample size limitation. He estimated the normal law parameters and their confidence intervals using the Probit method with maximum likelihood inference. The second raw of Table 4 shows his results calculated using uncertainty propagation. In the same way, the CIs on key velocities using the Brownian motion approach are evaluated by inspecting the plotted data in Fig. 5. The third row of Table 4 presents the retrieved results. The obtained results for the CIs limits and their equivalent deviations to the experimental ones are registered under the value and deviation label using each method.

Again, it is noted that the experimental estimation belongs to the determined confidence intervals. Moreover, the width of the CI intervals on V_1 and V_{99} is smaller using the Brownian motion approach. Effectively, the Brownian motion approach provides a CI equal to 104 ms^{-1} and 99 ms^{-1}, while the probit method estimates a CI of 180 ms^{-1} and 180 ms^{-1} for the V_1 and V_{99}, respectively. Moreover, the confidence intervals (CIs) estimated using the Brownian-based method are included in the those calculated using the Probit method. However, we remark that the CI of the V_{50} is wider than the equivalent estimation using the Probit method under the normality assumption of the perforation velocities. The interval width is 14.64 ms^{-1} and 23 ms^{-1} for the V_{50} using the probit and the Brownian-based approach, respectively. Finally, the deviation of the interval limits to the experimental estimation is lower using the Brownian-based approach for the V_1 and V_{99}. However, it is slightly higher for the V_{50}.

Thus, the proposed methodology slightly ameliorates the uncertainty on key velocities estimation even under the limiting hypothesis of constant diffusion and bullet deceleration. These results encourage deeper research for a better understanding and characterization of the process coefficients.

4. CONCLUSION

The Brownian motion approach is a recently proposed modelling method for ballistic resistance evaluation. In this first study, Goodness-of-fit tests, Chi-square and Kolmogorov-Smirnov criteria were implemented. The main limitation of this methodology is the need for large experimental databases. The likelihood inference is fundamental for applying the Brownian motion approach to feasible laboratory samples. For this purpose, the model has been tested on a large-sized experimental database and a subset of it. The model's estimated deceleration matches its prediction based on Rosenberg's formula. To further analyze the model results, the perforation probability curve is evaluated. Again, a good match was observed between the experimental estimation of the perforation probabilities and the numerically estimated ones using the large and small-sized databases. Moreover, the comparison of the stochastic model results with the Probit results confirms the competitive performance of the likelihood inference for the model's parameters estimation under a sample size restriction. In addition, based on the likelihood inference, it was observed that the Brownian motion approach slightly reduces the uncertainty on key velocities estimation compared to the Probit method. In contrast, the advantage of the stochastic model over existing methods is manifested by its ability to incorporate analytical formulation of the system physics. The bullet deceleration is included in this implementation while further characterization of the model's parameters may involve more physical parameters of the impact event.

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