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Novel Attitude Estimation of Strapdown Inertial Navigation Systems with Singular Value Decomposition Technique

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ABSTRACT

Davenport's q method & the Singular Value Decomposition (SVD) method are the two vigorous estimators that reduces Wahba's loss function. In these, the q method is slightly quicker due to its computation of optimum quaternion as an eigenvector of a symmetric 4x4 matrix through the prevalent eigenvalue. The ESOQ and ESOQ2 (EStimators of the Optimal Quaternion) and the QUEST (QUaternion ESTimator) algorithms are less determined as the extreme eigenvalue's distinguishing polynomial equation is solved by them. These estimators are apt to track the undulations of the sea with equivalent precision and accurateness. The SVD method is chosen and shown to be the most robust of all the hostile methods for the orientation of SDINS (Strap-Down Inertial Navigation Systems) using rate matching observations at sea in this paper. SVD is known most robust decomposition of all the decompositions of a matrix. SVD based attitude estimation being a batch technique would suffer from much less computational issues.

Keywords: Attitude estimation; Singular value decomposition; Transfer alignment; Moving launch platform; Critical error probability

NOMENCLATURE

L	-	Wahba loss function
y_i, f_i	-	Vectors in frame X and Y
X, Y	-	Set of vectors yi, fi
x_i	-	Weights of Wabha loss function
M	-	$\sum y_i f_i^T$
X_{opt}	-	Optimal direction cosine matrix
q	-	Optimal quaternion
λ_{max}	-	Maximum eign value
Ø	-	Non-linear function of eigen values
λ_0	-	First eigen value
Σ_{ii}	-	Singular values
σ^2	-	Variance

1. INTRODUCTION

SDINS (Strapdown Inertial Navigation Systems) are dead reckoning in nature. They need initial attitude estimates to start the navigation process. As such at sea, the initialisation of attitude is a tedious and time necessitating procedure that can be circumvented by employing an a priori aligned SDINS and transferring the alignment angles to a slave system through the weapon system. Such a procedure comes to be known as transfer alignment.

In this paper, we conduct a satisfactory transfer of alignment at sea using a priori aligned master system to the slave system using a batch of observations of angular velocity from both master and slave.

The conjecture is that master and slave together are rigid with reference to each other. We do so, with the highly arithmetically vigorous SVD (Singular Value Decomposition) technique for batch processing of data.

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2. LITERATURE REVIEW

The attitude interpretations are certainly epitomised as unit vectors in various spacecraft attitude systems. The entity vector in the sequence of the Earth's magnetic field & the unit vectors providing the course to the star or sun are the various instances.

In 1965, Grace Wahba[1] proposed a loss function for reckoning spacecraft attitude commencing from vector measurements, that are followed by all algorithms:

Resulting in the orthogonal matrix A by determinant "+1" that reduces the loss function is the Wahba's problem[1].

$$L(X) = \frac{1}{2}\Sigma_i \; x_i \left| y_i - Xf_i \right|^2 \quad (1)$$

where

$\{x_i\}$ indicates non-negative weights

$\{y_i\}$ indicates a set of unit vectors that are calculated in a spacecraft's body frame and

$\{f_i\}$ indicates the equivalent unit vectors in a reference frame.

For comparison of the Wahba's problem to Maximum Likelihood Estimation, we indicate the weights to remain transposed discrepancies, $x_i = \sigma_i^{-2}$.

For whom anticipated the weights regulated to unity, this choice contrasts from that of Wahba's and various authors.

Providing an outline of the most prevalent algorithms in a cohesive notation, and providing precision and speed evaluations is the theme of this paper. The efficacy of the anticipated algorithms will be evaluated in the actual launch scenario with rigid master and slave configuration on ship-launched weapon systems.

Emphasis is exposed as to why attitude information is critical to the start of inertial navigation. Paper[16] explores robust strategies for in-motion inertial navigation explaining various statistically robust methods. DARPA, US is relying on MEMS technology for positing of small vehicles, for which attitude information shall become necessary[18]. The need for ubiquitous inertial navigation is given in Stovall[19].

Attitude is represented severally, viz., rotation vector, Rodriguez parameters, DCM and the quaternion. While other representations suffer from gimbal lock problem, the quaternion representation overcomes the lacunae and is considered the most robust form of attitude representation. Jim Wu *et. al.*[15] give the angular velocity of the quaternion representation by successive differentiation method. Jim Wu *et. al.*[25-26] further provide variants to the angular velocity differential equation using the quaternion. The angular velocity vector is captured by the gyros. Thus a quaternion propagation differential equation is provided by Jim Wu in their papers. This is very useful in solving the Wahba's problem, as Davenport method uses quaternions instead of the DCM's to solve the Wahba's problem.

3. ORTHOGONAL PROCRUSTES PROBLEM

The Wahba's loss function will be given as

$$\mathrm{L}(\mathrm{X}) = \Sigma_i \; x_i - tr\left(XY^T\right) \tag{2}$$

With

$$Y = \Sigma_i \; x_i y_i f_i^T \tag{3}$$

Hence, $\mathrm{L}(\mathrm{X})$ can be reduced while the trace, XY^T, is exploited.

To discover the orthogonal matrix X that is flanking to Y in the sense of the Frobenius norm, i.e., similar to Orthogonal Procrustes problem.

$$\left|M\right|_F^2 = \Sigma_{ij} \; M_{ij}^2 = tr\left(MM^T\right) \tag{4}$$

Now

$$\| X - Y \|_F^2 = \| X \|_F^2 + \| Y \|_F^2 - 2tr(XY^T) = 3 + \| Y \|_F^2 - 2tr(XY^T) \tag{5}$$

Hence, by the precondition that the determinant of A must be $+1$, both the Wahba's problem [1] and the orthogonal Procrustes problem are similar.

4. FIRST SOLUTIONS

The first solution of Wahba's problem, according to J. L. Farrell and J.C. Stuelpnagel [2] is, any real matrix including Y, has the polar decomposition

$$Y = OS \tag{6}$$

where

O indicates orthogonal,

S indicates positive semi-definite and symmetric.

Then S can be diagnolised by

$$S = VDV^T \tag{7}$$

where

V indicates orthogonal matrix,

D indicates transverse with components organised in reducing order.

Now, the optimum attitude estimate can be specified as

$$X_{opt} = OVdiag[1 \quad 1 \quad \det O]V^T \tag{8}$$

Mostly but not assured always, $X_{opt} = O$ whereas $\det O$ is positive.

The alternate solution proposed by R. H. Wessner is given:

$$X_{opt} = (Y^T)^{-1}(Y^T Y)^{(1/2)}, \tag{9}$$

i.e., similar to

$$X_{opt} = Y(Y^T Y)^{-1/2} \tag{10}$$

Necessitating Y to be non-singular is the detriment having with Equations (9) and (10) i.e., even though two vector interpretations are adequate to conclude the attitude, still, minimum three vector interpretations[7] are required to visualise the pseudo inverse solution.

The various clarifications to Wahba's problem are also provided by R. Desjardins, J. E. Brock[5], J. R. Velman and Wahba [1].

5. UNCONSTRAINED LEAST-SQUARES

Without necessitating the orthogonality constraint, there is a chance of reducing Wahba's loss function i.e., by

$$X_{unconstrained} = Y(\Sigma_i x_i f_i f_i^T)^{-1}. \tag{11}$$

This signifies

$$Y = X_{unconstrained} = (\Sigma_i x_i f_i f_i^T) \tag{12}$$

Here $X_{unconstrained}$ is merely approximately orthogonal, hence it's not similar to polar decomposition even though it seems equivalent. Brock[5] proposed a solution that is analysed by Bar-Itzhack and Markley[8].

6. DAVENPORT'S Q METHOD

A genuine innovation came when Wahba's problem to spacecraft attitude determination was modelled by Paul

Davenport in search of a quaternion-based solution for the attitude estimation[10-11].

X can be parameterised by a unit quaternion[8-9, 28]

$$q=\begin{bmatrix} q \\ q_4 \end{bmatrix}, \text{ where } |q|^2=1, \tag{13}$$

as

$$X=(q_4^2-|q|^2)I+2qq^T-2q_4[qX] \tag{14}$$

The homogenous quadratic function of q can be composed as,

$$tr(XY^T)=q^TKq \tag{15}$$

where K denotes symmetric traceless matrix[4]

$$K\equiv\begin{bmatrix} S-ItrY & z \\ z^T & trY \end{bmatrix} \tag{16}$$

With

$$S=B+BT \tag{17}$$

And

$$z\equiv\begin{bmatrix} Y_{23}-Y_{32} \\ Y_{31}-Y_{13} \\ Y_{12}-Y_{21} \end{bmatrix}=\Sigma_i x_i y_i Xf_i \tag{18}$$

Hence, the standardised eigenvector of K can be proved by the largest eigenvalue i.e., the result of Equation (19) is optimal unit quaternion[24].

$$K_{q_{opt}}=\lambda_{\max}q_{opt} \tag{19}$$

For solving the symmetric eigenvalue problem, many robust algorithms exist. [22] They can be implemented easily in MATLAB. If the two prevalent eigenvalues of K are identical then there is no solution. The data aren't abundant to conclude the attitude distinctively, i.e., not a catastrophe of the q method. It is the absence of sufficient data to conclude the estimation process.

7. QUATERNION ESTIMATOR (QUEST)

Equation (19) is comparable to the below given Equation (20) and Equation (21)[14,12]

$$[(\lambda_{\max}+trY)I-S]q=q_4Z \tag{20}$$

and

$$(\lambda_{max}-trY)q_4=q^TZ \tag{21}$$

Equation (20) provides

$$q=q_4[(\lambda_{\max}+trY)I-S]^{-1}z$$
$$=q_4\{adj[(\lambda_{\max}+trY)I-S]z/\det[(\lambda_{\max}+trY)I-S]\} \tag{22}$$

For a general 3x3 matrix, the Cayley-Hamilton theorem G states that

$$q=q_4[(\lambda_{\max}+trY)I-S]^{-1}z=q_4\{adj[\lambda_{\max}+trY)I-S]z/det[(\lambda_{\max}+trY)I-S]\}$$

$$G^3-(trG)G^2+[tr(adjG)]G-(detG)I=0 \tag{23}$$

where $adjG$ is the typical adjoint (adjugate) of G. Hence the adjoint can be conveyed as

$$adjG=G^2-(trG)G+[tr(adjG)]I \tag{24}$$

In precise

$$adj[\lambda_{\max}+trY)I-S]=\alpha I+\beta S+S^2 \tag{25}$$

where

$$\alpha\equiv\lambda_{\max}^2-(trY)^2+tr(adjS) \tag{26}$$

and

$$\beta\equiv\lambda_{\max}-trY \tag{27}$$

We also enunciate

$$\gamma\equiv\det[(\lambda_{\max}+trY)I-S]=\alpha[(\lambda_{\max}+trY)-\det S] \tag{28}$$

The optimal quaternion can be specified as

$$q_{opt}=\frac{1}{\sqrt{\gamma^2+|x|^2}}\begin{bmatrix} x \\ \gamma \end{bmatrix}, \tag{29}$$

where

$$x\equiv(\alpha I+\beta S+S^2)z \tag{30}$$

maximum Eigen value $\lambda_{\max}$ plays vital for all these computations whereas it can be attained by switching Equation (22) into Equation (21), which produces the equation:

$$0=\psi(\lambda_{\max})\equiv\gamma(\lambda_{\max}-trY)-z^T(\alpha I+\beta S+S^2)z \tag{31}$$

A fourth-order equation for λmax can be attained by switching Equations (26–28). This can be cracked rationally by using the distinctive equation $\det(K-\lambda_{\max}I)=0$. Conversely, that $\lambda_{\max}$ is precisely close to

$$\lambda_0\equiv\Sigma_i x_i \tag{32}$$

if the enhanced loss function

$$L(A_{opt})\equiv\lambda_0-\lambda_{\max} \tag{33}$$

is solved by the Newton-Raphson iteration method, $\lambda_{\max}$ can be effortlessly attained, beginning from $\lambda=0$ as the primary estimate. Statistically, a distinct rehearsal is mostly ample.

Nevertheless, one of the ways to find eigenvalues is elucidating the specific equation, commonly, Davenport's original q method is much robust than QUEST in principle i.e., distinguished statistically. Equation (29) doesn't describes the optimal quaternion, if

$$\gamma^2+|x|^2=0, \tag{34}$$

Therefore, the technique of consecutive cycles to lever this circumstance is contrived by Shuster[10-11]. These are slightly lavish computationally as for regulating the number of consecutive cycles accomplished accurate norm is preferred. Switching Equation (30) into Equation (34) and changing the Cayley-Hamilton theorem twice to exclude S^4 and S^3

contributes, and after tedious algebra,

$$\gamma^2 + |x|^2 = \gamma(\frac{d\psi}{d\lambda}) \ , \tag{35}$$

Where Equation (31) implicitly defines $\psi(\gamma)$, the quartic function. For the Newton-Raphson iteration used for $\lambda_{\max}$ to be efficacious, $\frac{d\psi}{d\lambda}$ is to be invariant underneath cycles, and this capacity must be non zero. Therefore $(q_{opt})_4 = 0$ and the optimal attitude exemplifies a 180° cycle which specifies that the singular condition of Equation (34) is perceived to be correspondent to $\gamma = 0$. To get a γ, we can always run consecutive cycles of iteration such that

$$(q_{opt})_4 > q_{\min} \tag{36}$$

For any $q_{\min}$ in $(0, 1/2)$, by asserting that

$$\gamma > q^2_{\min}(d\psi / d\lambda) \tag{37}$$

To elude loss of numerical precision in the computation, $q_{\min} = 0.1$ is ample in preparation.

An appraisal of covariance of the rotation slant error vector in the body frame is also provided by Shuster[3],

$$P = [\Sigma_i x_i (I - y_i y_i^T)]^{-1} \tag{38}$$

And supposing Gaussian measurement errors, disclosed that the optimised loss function $L(X_{opt})$ observes a chi-square probability distribution to a worthy calculation. QUEST, in 1979, initially smeared in the MAGSAT mission, is the most commonly used algorithm for Wahba's problem.

8. SINGULAR VALUE DECOMPOSITION (SVD) METHOD

The matrix Y has the SVD (Singular Value Decomposition)[13].

$$Y = U\Sigma V^T = U diag[\Sigma_{11} \quad \Sigma_{22} \quad \Sigma_{33}]V^T \tag{39}$$

where U and V are orthogonal, and the particular tenets follow the discriminations $\Sigma 11 \geq \Sigma 22 \geq \Sigma 33 \geq 0$.

Then

$$tr(AB^T) = tr(AV diag[\Sigma_{11} \quad \Sigma_{22} \quad \Sigma_{33}]U^T$$

$$= tr(U^T AV diag[\Sigma_{11} \quad \Sigma_{22} \quad \Sigma_{33}]) \tag{40}$$

For A to be a orthogonal rotation matrix, $\det A = 1$, and the optimal direction cosine matrix can be given as

$$U^T A_{opt} V = diag[1 \quad 1 \quad (det U)(det V)] \tag{41}$$

which contributes the ideal attitude matrix:

$$A_{opt} = U \ diag \ [1 \ 1 \ (detU)(detV)] \ V^T$$

$$A_{opt} = U \ diag \ [1 \ 1 \ (detU)(detV)] \ V^T \tag{42}$$

Equation (42) is indistinguishable from Equation (8) with $U = WV$, subsequently, the novel result by Farrell and Stuelpnagel [2] corresponds to the SVD solution. The variance is that SVD algorithms be existent now.

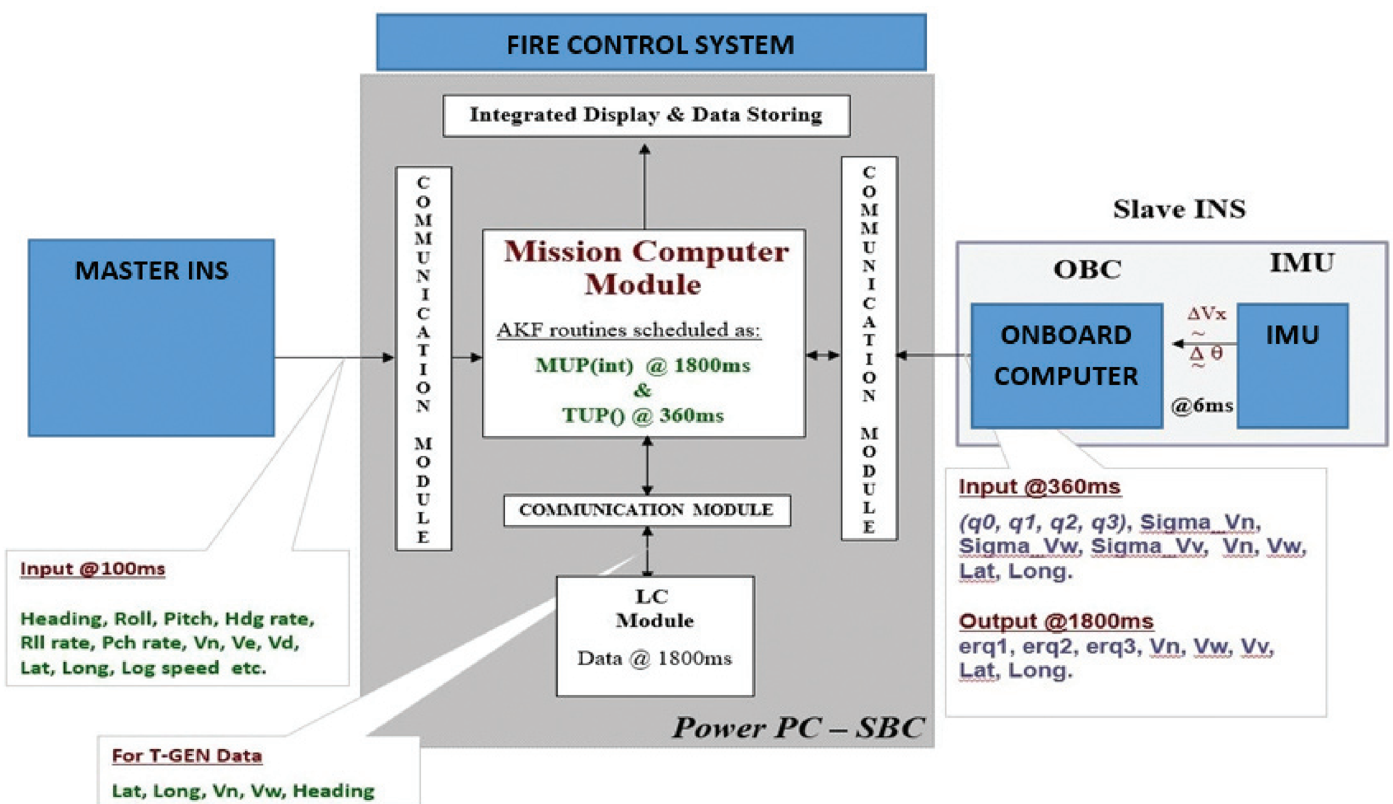

Figure 1. Existing TA Software on FCS.

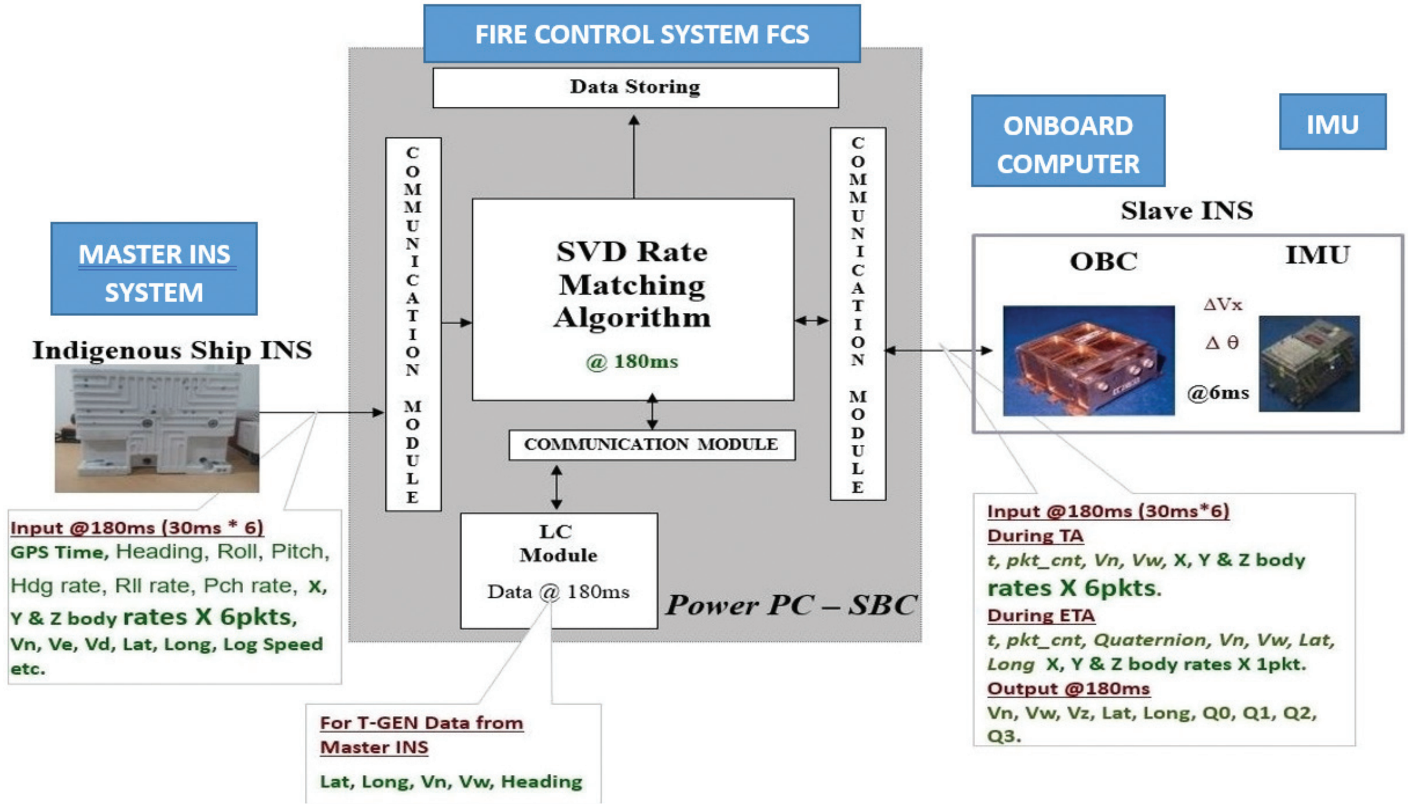

Figure 2. Proposed TA software on FCS.

It is expedient to delineate

$$S_1 \equiv \Sigma_{11}, S_2 \equiv \Sigma_{22} \text{ and}$$

$$S_3 \equiv (\det U)(\det V)\Sigma_{33} \quad (43)$$

so that S_1 S_2 $|S_3|$. The error covariance of attitude is specified as

$$P = U diag[(S_2+S_3)^{-1}(S_3+S_1)^{-1}(S_1+S_2)^{-1}]U^T \quad (44)$$

The singular values are allied to the eigenvalues of Davenport's K matrix,

$$\lambda_{max} \equiv \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \quad \text{by}$$

$$\lambda_1 = S_1 + S_2 + S_3$$

$$\lambda_2 = S_1 - S_2 - S_3$$

$$\lambda_3 = -S_1 + S_2 - S_3$$

$$\lambda_4 = -S_1 - S_2 + S_3 \quad (45)$$

The eigenvalues summation to zero as K is trace-less. The condition of vast covariance i.e., the peculiarity condition, is

$$S_2 + S_3 = 0 \quad (46)$$

This is comparable to the previously-stated unnoticeable condition for Davenport's q method i.e.,

$$\lambda_1 = \lambda_2$$

9. REQUIREMENT OF TIME SYNCHRONISATION

As the body rates are compared there is a tight synchronisation requirement amongst Master and Slave INS through OBC.

This has been conquered using a specified protocol formation for command and response including data transmission among FCS (where Filter is executed), Master INS, and Slave INS through OBC.

We have done alignment of SDINS at sea for the approximation of master to slave misalignment angles using the batch mode SVD technique in the ship.

10. INSTRUMENT TEST SETUP AT SEA

As exposed in Fig. 1, in the old transfer alignment scheme, the master and slave's velocity information is brought to the FCS computer, wherein the velocities can be equated with the assistance of a suitable Kalman filter and the error estimates in attitude are computed[27]. However, the major issue with such a scheme is the necessity of manoeuvres for active convergence of azimuth solution. This can be alleviated if we resort to the comparison of angular velocities under suitable excitation of roll and yaw manoeuvres imposed on the launcher platform before the lift-off.

In Fig. 2, the block diagram displays the setup of the new transfer alignment scheme. The master INS and the IMU supply the incremental angles data, captured via the On-Board Computer at the Fire Control System.

The SVD based algorithm operates on the data existing both from master and slave to arrive at the optimal solution of the attitude of the IMU with respect to the master. The master also supplies the attitude information with respect to the ground. The interalia estimated misalignment is coupled

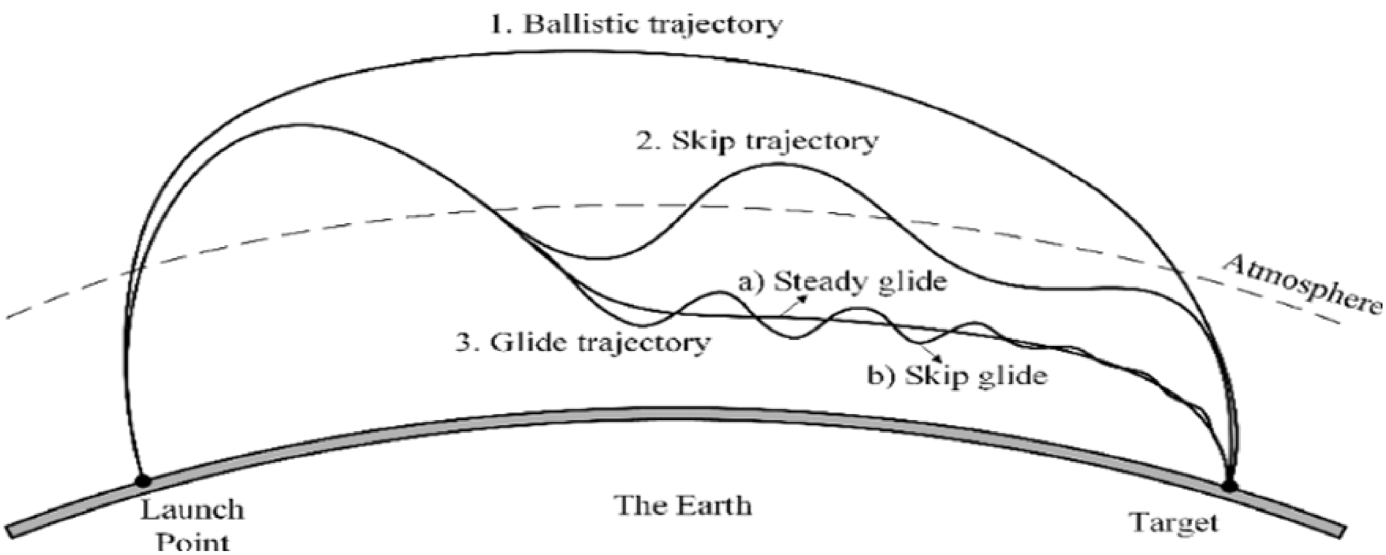

Figure 3. Implicit missile trajectory.

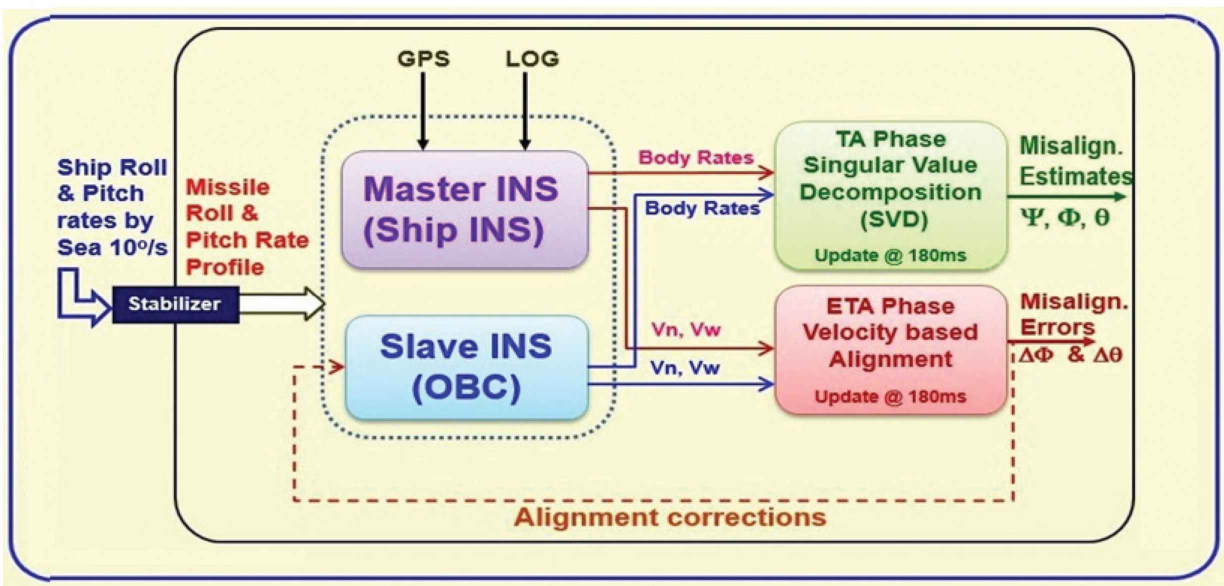

Figure 4. Rate matching based TA scheme.

with the master to ground attitude information to arrive at the IMU to ground attitude estimate.

When compared vis-à-vis the KF based technique, the SVD based technique is able to arrive at the attitude with an accuracy atleast two orders enhanced than the attitude obtained by using the KF based method. Also the need for orientation of ship to be altered for the determination of alignment is also alleviated with the usage of SVD based solution. The cross range accuracy obtained with the method is improved immensely as evidenced by the resultant data.

Figure 3 shows the proposed trajectory modelling of the flight over a downrange of the weapon system. The flight path is modelled to increase the range by conversion of KE-PE energy profile during the flight.

Figure 4 shows the integration of both SVD and ETA schemes for the evaluation of yaw, roll, and pitch respectively, and the integration is conducted in the FCS computer.

Figures 5 and Fig. 6 show the convergence of roll, pitch yaw angles under excitation of the roll and yaw channels. The velocities are converged to within $(0.01\ \mathrm{m/sec})(0.01\ \mathrm{m/sec})$ in both North and East channels. Figure 6 shows the estimation of the constant misalignment angle between the master and slave in rigid mounting conditions. The SVD method while employed in the rigid body consideration, has full potential to even align non-rigid systems. However, the non-rigid formulation of SVD method is deferred herewith for an appropriate use case occasion whence non-rigid SVD formulation could be employed.

11. RESULTS

The proof of convergence of the SVD filter must reflect in the convergence of velocities of slave and master. To this end, Fig. 5 depicts results to show the eventual convergence of velocity information and remaining steady ever after. Since the master and slave are rigidly mounted, the misalignment angles between master and slave must remain a random constant. Figure 6 shows the constant behaviour of the estimated quantities. The roll, pitch and yaw channels are shown in Fig. 6.

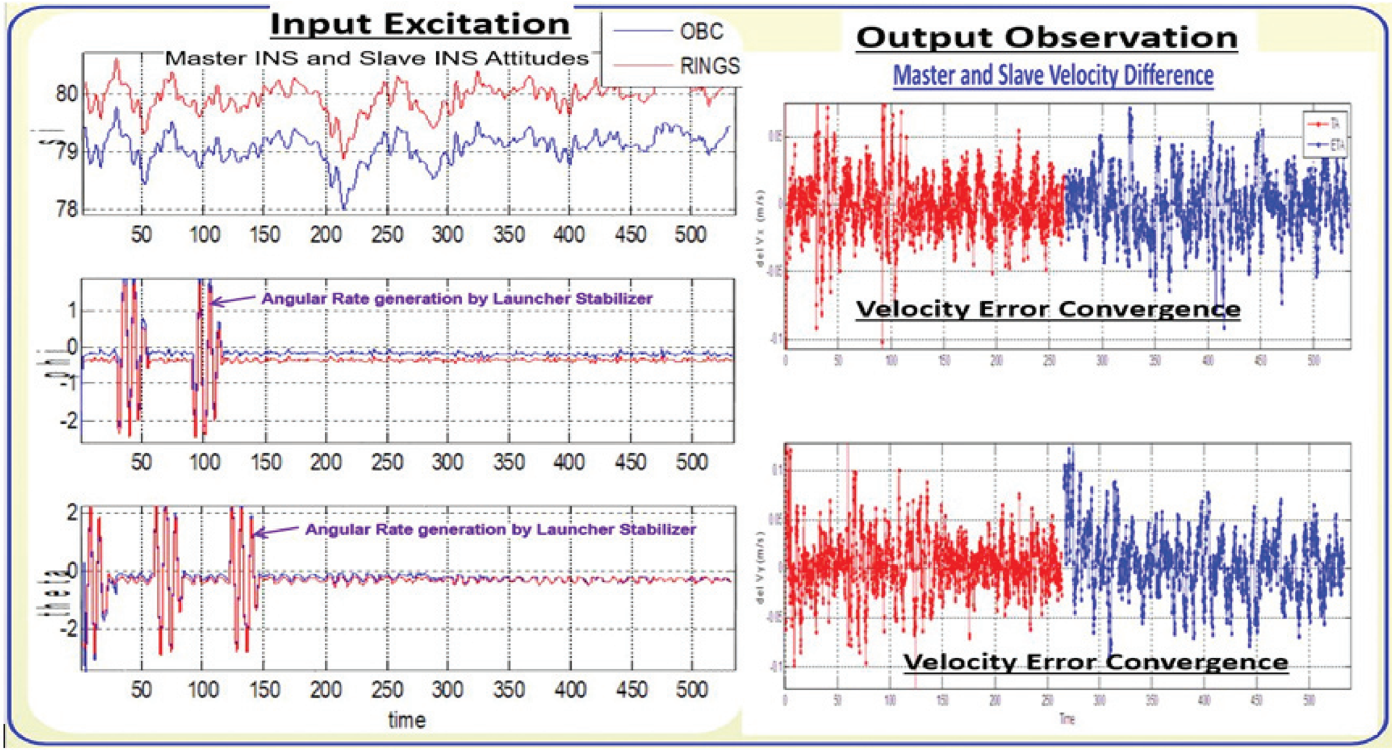

Figure 5. Misalignment estimates and observations.

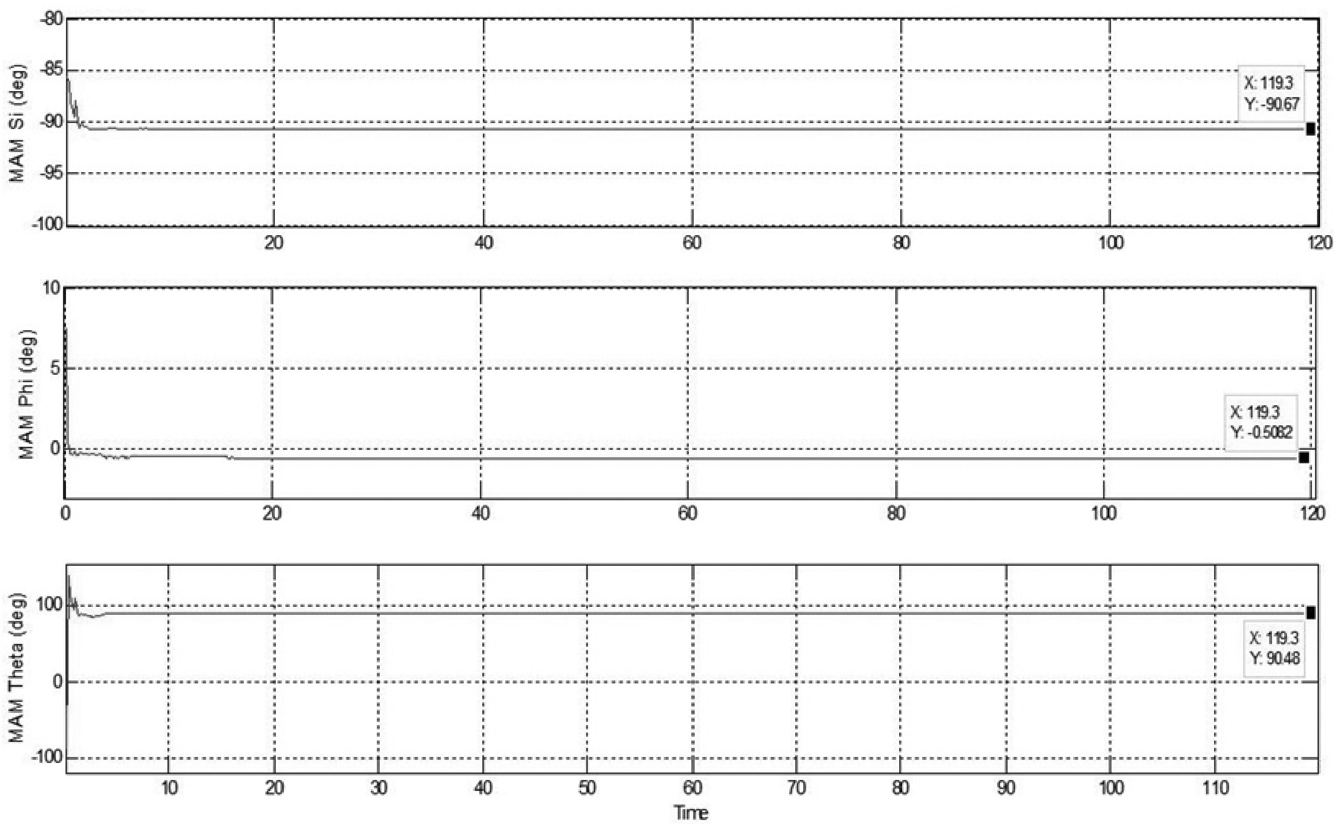

Figure 6. SI, PHI, THETA misalignment estimation.

They are sturdy constant. Figure 7 illustrates the repeatability of several runs of the SVD based attitude estimation technique at sea. The estimation of azimuth has a typical deviation of within $0.1^0\,(3\sigma)$ while roll and pitch have far less variance as compared to azimuth channel as given in the table.

The position errors at the end of time-of-flight runs are about 1km CEP (3σ).

In actual trials, the cross-range error was found to be well with the 0.142 per cent of the downrange CEP (1σ).

Run No	Ψ_{MAM}	Φ_{MAM}	θ_{MAM}	Attitude Rates Rll & Pch (°/s)	Profile time (min)	X_{err} (m)	Y_{err} (m)	Z_{err} (m)
1	-90.67	-0.50	90.48	±2, ±1.8	2	1480	-8	51
2	-90.72	-0.47	90.44	±2, ±1.8	2	453	612	32
4	-90.72	-0.51	90.33	±2, ±1.8	2	859	866	34
5	-90.72	-0.48	90.32	±2, ±1.8	2	-71	-1112	-14
6	-90.73	-0.46	90.37	±2, ±1.8	2	924	-1082	-32
7	-90.71	-0.38	90.40	±2, ±1.8	2	-622	-521	-51
8	-90.75	-0.51	90.34	±2, ±1.8	2	4	800	10
9	-90.77	-0.50	90.36	±2, ±1.8	2	1125	261	-25
10	-90.76	-0.47	90.33	±2, ±1.8	2	17	-512	-60
11	-90.78	-0.51	90.36	±2, ±1.8	2	-240	529	-87
12	-90.78	-0.46	90.35	±1.2, ±0.8	3	-275	203	-112

Figure 7. Results using New TA technique.

12. CONCLUSION

SVD based attitude algorithm has developed and also been tested on the ship. Very decent accuracy has been obtained in roll, pitch, & yaw channels. The plots display the convergence of roll, pitch & yaw channels up to a precision of 0.02 degrees.

The accuracy is sufficient to bring the cross-range within 0.14% of the downrange capability of the weapon, while the GPS aided error will further reduce the cross-range within a few meters.

SVD aided rate matching technique is most suitable for ship-launched weapon systems as the sea undulations aid in the convergence of the attitude misalignment angles without the need for additional manoeuvres.

13. FUTURE WORK

The algorithm assumes rigidity between master and slave for the SVD batch processing to work. However, minute movements between master and slave could never be alleviated in true mechanical launch scenarios.

The present strategy may be improvised to include the play inter-alia between the master and the slave, therewith rendering the algorithm to work in all deployable scenarios.

The other challenge not considered in the present paper is the coupling of inferior grade gyros with the high-grade master gyros employed in the present scenario.

The performance of transfer alignment with degraded gyros, if when improved would be useful in employing the SVD technique for short-range missile systems. Such an attempt is currently underway in the upcoming ship-to-air launch missions from on-board ships.

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