Defence Science Journal, Vol. 72, No. 2, March 2022, pp. 227-235, DOI : 10.14429/dsj.72.16933

The Complete Analytical Solution of the TDOA Localization Method

Petr Hubáček, Jiří Veselý, and Jana Olivová*

Department of Communication Technologies, Electronic Warfare and Radars, University of Defence in Brno, Brno - 66210, Czech Republic

**E-mail: jana.olivova2@unob.cz*

ABSTRACT

This article is focused on the analytical solution of a Time Difference of Arrival (TDOA) localization method, including analysis of accuracy and unambiguity of a target position estimation in 2D space. The method is processed under two conditions - sufficiently determined localization system and an overdetermined localization system. It is assumed that the TDOA localization system operates in a LOS (Line of Sight) situation and several time-synchronized sensors are placed arbitrarily across the area. The main contribution of the article is the complete description of the TDOA localization method in analytical form only. It means, this paper shows a geometric representation and an analytical solution of the TDOA localization technique model. In addition, analyses of unambiguity and solvability of the method algorithm are presented, together with accuracy analysis of this TDOA technique in analytical form. Finally, the description of this TDOA method is extended to an overdetermined TDOA system. This makes it possible to determine and subsequently optimize its computational complexity, for example increase its computational speed. It seems that such a description of the TDOA localization technique creates a simple and effective tool for technological implementation of this method into military localization systems.

Keywords: Time Difference of Arrival; Localization method; Covariance matrix; Target position estimation; Electronic Intelligence

1. INTRODUCTION

The localization of emitter of a non-cooperative signal is a common task in many security and military surveillance systems. These systems can be applied to a wide variety of areas, such as electronic warfare applications[1], target tracking, electronic intelligent systems, perimeter protection systems[2] or location-based services[3]. The most commonly used approaches, or techniques, for measuring a target position are Time of Arrival (TOA)[4], Time Difference of Arrival[5-6], Received Signal Strength (RSS)[7], Doppler Difference (DD)[8], Angle of Arrival (AOA)[9] and combinations of these techniques[10-11]. The localization techniques can be described by a number of different characteristics, such as accuracy, implementation complexity, cost, etc. The TDOA localization systems belong to the group of complex techniques.

In this paper, we focus on the TDOA localization system. The principle of operation of such system is based on measuring the Time Difference of Arrival (Standard TDOA) or the Time of Arrival (Leading Edge TDOA, LE – TDOA) of the received signal on different receiver positions. The LE – TDOA system will be considered in this article. Specifically, the system will contain N sensor nodes (receiving stations) whose positions are known, and each sensor node will be able to measure time of arrival of the target signal and all sensors will be time-synchronized. If the number of sensors is greater than N+1 in N dimensional space, then the TDOA system is so-called overdetermined. Once the measured data is obtained, the range difference between the target and two different receiving stations can be calculated. In this connection, a set of equations can be obtained and the coordinates of the target in the Cartesian coordinate system are the solution of these equations. It is clear that determination of an accurate target location requires an effective algorithm for its calculations. Many processing algorithms, with different complexity and restrictions, have been proposed for location calculation based on TDOA measurement. A group of methods that are commonly used to simplify the location computation is based on a linearization of these equations, for example Taylor Series expansion. The main idea of the Taylor Series Method is to expand the first Taylor Series by nonlinear equations at the initial estimation of the target position, and then, solve the equations by iteration[12-13]. In many cases, this linearization does not introduce errors in the position determination. However, the linearization can introduce significant errors when an initial position is wrongly estimated. Modifications of these techniques are the Least Squares Method (LS) or Weighted Least Squares (WLS) Method[14-15]. These techniques can achieve Maximum Likelihood (ML) estimate that maximizes the probability of each particular position estimate being a true position location. A second group of methods, used to solve nonlinear equations, includes "closed-form" methods. These methods don't need an initial estimation of the target's location and iterations are not necessary either. For example, the technique[16-17] transforms nonlinear equations into pseudo-linear equations by introducing auxiliary variables. Then, the equations are solved by two-step

Received : 18 March 2021, Revised : 07 September 2021
Accepted : 27 September 2021, Online published : 11 May 2022

WLS or Least Squares (LS) Method. An exact solution of nonlinear equations has been provided for arbitrarily placed receiving stations in a sufficiently determined system[18-19].

There are a large number of different analytical methods solving these equations. They can be divided into three main groups. One group are methods using trigonometric functions[20], second group is using matrix operations[21] and third group uses direct algorithm with coordinate system transformation. Unfortunately, they are quite complicated to completely describe all the properties of the TDOA method. Therefore, it is necessary to develop a robust tool for a complete analytical description of the TDOA localization technique. This goal can be accomplished using the last-mentioned algorithm, i.e. a direct algorithm. This paper is organized as follows: Section 2 introduces a geometric representation of the TDOA method, a basic model of TDOA measurement and its analytical solution. Section 3 shows accuracy analysis of the method and analysis of unambiguity and solvability of the TDOA algorithm. Section 4 describes analytical solution of hyperbolic equations for an overdetermined TDOA system. Section 5 briefly presents advantages of the proposed method in practical applications.

2. GEOMETRIC REPRESENTATION AND ANALYTICAL SOLUTION OF THE TDOA LOCALIZATION TECHNIQUE MODEL

First, a typical scenario for using the TDOA localization method is given in this section. Furthermore, there is a mathematical model of the method and its proposed an analytical solution.

Assumed is a network composed of a set of fixed position and time-synchronized receiving sensors. In this example, there are three sensors S_1 ... S_3, and one target T. The number of sensors N+1 = 3 represents sufficiently determined system in 2D. The arrangement of such a network is shown in Fig. 1.

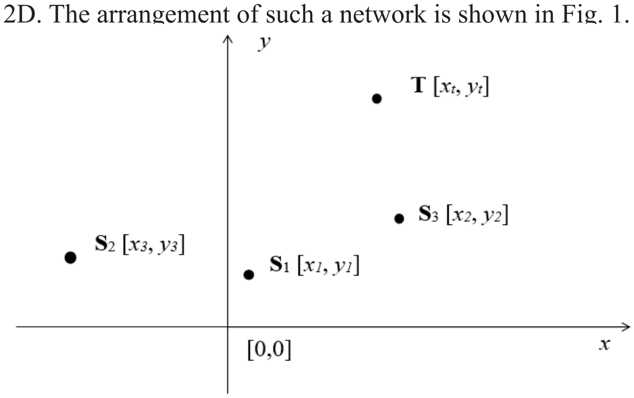

Figure 1. Sensor network and target arrangement.

The time of arrival (TOA_i) for the received signal measured by the sensor i is

$$TOA_i = t_0 + \frac{\sqrt{(x_i - x_t)^2 + (y_i - y_t)^2}}{c_l} \tag{1}$$

where t_0 is the unknown initial transmission time (time of transmitting the signal by the target), x_i, y_i are the sensor coordinates, x_t, y_t are the target coordinates and c_l is the speed of electromagnetic waves. Then, the measured TOAs to particular sensors are written according to (1) as:

$$TOA_1 = t_0 + \frac{\sqrt{(x_1 - x_t)^2 + (y_1 - y_t)^2}}{c_l} \tag{2}$$

$$TOA_2 = t_0 + \frac{\sqrt{(x_2 - x_t)^2 + (y_2 - y_t)^2}}{c_l} \tag{3}$$

$$TOA_3 = t_0 + \frac{\sqrt{(x_3 - x_t)^2 + (y_3 - y_t)^2}}{c_l} \tag{4}$$

This set of non-linear equations (2), (3) and (4) represents the model of the TDOA localization method.

Generally, in a standard coordinate system $\{x, y\}$, the arrangement of the sensors and the target can be arbitrary. In order to simplify the analytical solution of these nonlinear equations, it is advisable to transform this arrangement into a new coordinate system $\{x^*, y^*\}$ by translation and rotation via the following transformation equations:

$$x^* = (x_t - x_1).\cos(\alpha) + (y_t - y_1).\sin(\alpha) \tag{5}$$

$$y^* = -(x_t - x_1).\sin(\alpha) + (y_t - y_1).\cos(\alpha) \tag{6}$$

where $\alpha = \tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$.

Then, the new sensor coordinates are $\mathbf{S}_1^*[0,0]$, $\mathbf{S}_2^*[a,0]$, $\mathbf{S}_3^*[b,c]$ and the new target coordinates are $\mathbf{T}^*[x_t^*, y_t^*]$. This situation is shown in Fig. 2.

Then, a new system of hyperbolic equations is obtained that is equivalent to (2), (3), (4) after the transformation:

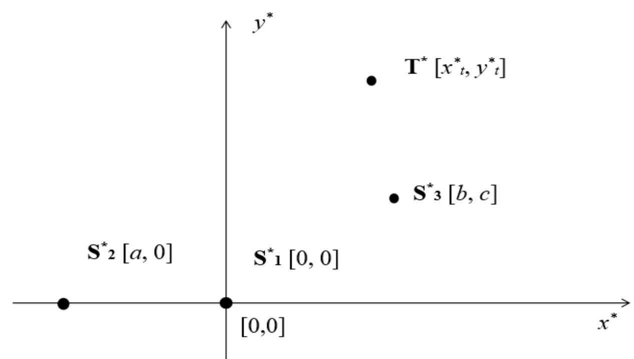

Figure 2. Sensor network and target arrangement in the new coordinate system.

$$TOA_1 = t_0 + \frac{\sqrt{x_t^{*2} + y_t^{*2}}}{c_l} = t_0 + \frac{K}{c_l} \tag{7}$$

$$TOA_2 = t_0 + \frac{\sqrt{(a - x_t^*)^2 + y_t^{*2}}}{c_l} \tag{8}$$

$$TOA_3 = t_0 + \frac{\sqrt{(b - x_t^*)^2 + (c - y_t^*)^2}}{c_l} \tag{9}$$

where $K^2 = x_t^{*2} + y_t^{*2}$.

By subtracting equation (7) from equations (8), and (9) and by entering substitutions $L = (TOA_2 - TOA_1).c_l$,

$R = \left(TOA_3 - TOA_1\right).c_l$ we get the following formulas:

$$L = \sqrt{\left(a - x_t^*\right)^2 + y_t^{*2}} - K \quad (10)$$

$$R = \sqrt{\left(b - x_t^*\right)^2 + \left(c - y_t^*\right)^2} - K \quad (11)$$

By algebraic reduction, the formula for the x^*_t target coordinate is following:

$$x_t^* = A + B.K \quad (12)$$

where $A = \frac{a - L^2}{2.a}$ and $B = \frac{-L}{a}$.

Similarly, the formula for the y^*_t target coordinate is following:

$$y_t^* = C + D.K \ (13)$$

where $\mathrm{C} = \frac{\mathrm{b}^2 + \mathrm{c}^2 - 2.\mathrm{a.A} - \mathrm{R}^2}{2.\mathrm{c}}$ and $\mathrm{D} = \frac{-\mathrm{R} - \mathrm{b.B}}{\mathrm{c}}$

.

Substituting equations (12) and (13) into K^2 leads to:

$$K^2 = x_t^{*2} + y_t^{*2} = \left(A + B.K\right)^2 + \left(C + D.K\right)^2 \quad (14)$$

Then, the roots of quadratic equation (14) $K_{1,2}$ are:

$$K_{1,2} = \frac{-N \pm \sqrt{N^2 - 4.M.P}}{2.M} \quad (15)$$

where $N = A.B + C.D$, $M = B^2 + D^2 - 1$ and $P = A^2 + C^2$.

Finally, the target coordinates x_t^*, y_t^* in the transformed coordinate system $\{x^*, y^*\}$ can be determined by substituting the roots of equation (15) back into formulas (12) and (13). The transformation of the target position back into the standard coordinate system $\{x, y\}$ is provided by the following formulas:

$$x_t = x_t^*.\cos\left(\alpha\right) - y_t^*.\sin\left(\alpha\right) + x_1 \quad (16)$$

$$y_t = x_t^*.\sin\left(\alpha\right) + y_t^*.\cos\left(\alpha\right) + y_1 \quad (17)$$

From a practical point of view, the derived algorithm of the TDOA method is implemented in the following way. Firstly, the TOA_1 to TOA_3 are measured. Secondly, the parameters L and R are determined. Then, all the variables A, B, C, D, M, N and P are computed. Consequently, the roots of quadratic equation $K_{1,2}$ can be determined. Note, the roots $K_{1,2}$ represent distances of the possible targets **T** to the sensor $\mathbf{S}_1$. Finally, the target coordinates x_t, y_t are found.

3. ACCURACY AND SOLVABILITY ANALYSIS OF THE TDOA TECHNIQUE

3.1 TDOA Solvability and Unambiguity Analysis

The model of the TDOA localization technique, expressed by the set of equations (2), (3) and (4), unambiguously assigns an arbitrary point (target) from the Cartesian coordinate system $\{x, y\}$ to the hyperbolic coordinate system $\{(TOA_2 - TOA_1), (TOA_3 - TOA_1)\}$. However, a reverse mapping from the hyperbolic plane to the $\{x, y\}$ plane is ambiguous. It is clear from the quadratic term (15) appearing in the analytical solution of TDOA equations. Thus, the proposed algorithm can generally lead to four different solutions:

- Both roots $K_{1,2}$ of the quadratic equation (15) are real positive numbers, i.e. we can compute coordinates of two different real targets (a real target is a target with real coordinates in $\{x, y\}$ plane);
- The two roots $K_{1,2}$ of the quadratic equation are two identical real positive numbers, i.e. we can compute coordinates of one real target;
- The two roots $K_{1,2}$ of the quadratic equation are two complex numbers, i.e. we cannot determine coordinates of the target. This is an example of a non-real target;
- One root is positive and the other root is negative, i.e. we can compute only coordinates of a real target ($K > 0$). The target with $K < 0$ is physically meaningless as K can never be negative (due to it represents a distance).

The following simulation provides an example of unambiguity and solvability analysis of the proposed technique. The following sensor network is assumed. It composed of 3 receiving sensors with coordinates $\mathbf{S}_1$[0,0], S_2[-7000m,0], S_3[7000m,5000m] and the target at position T[10000m,50000m]. If the TOAs are determined and subsequently used as inputs for the proposed algorithm, then, the algorithm returns two target positions T_1[10000m,50000m] and T_2[-14995m, 42538m], where T_1 is "true" real target position and T_2 is "false" real target position. This result shows that the algorithm works from the un-ambiguity point of view.

Next, the TDOA algorithm can be analysed from the solvability point of view. Taking into consideration the proposed algorithm derived in Section 2, it can be easily stated that it does not have any solution under the three following conditions:

- $a = 0$, in Equation (12),
- $c = 0$, in Equations (13),
- $M = 0$, in Equation (15).

The first condition is satisfied only when the receiving sensor $\mathbf{S}_2$ has the same coordinates as the sensor $\mathbf{S}_1$. It means that the TDOA system has only two receiving stations and the coordinates of the target cannot be determined.

The second condition is met only if the receiving station $\mathbf{S}_3$ has coordinates [b,0]. This means that all three sensors are collinear, i.e. they are all in alignment. However, this situation can be solved by a relatively simple modification of the proposed algorithm. Analysis of the third condition is more complicated than the previous analyses. Thus, the following approach should be implemented. The condition $M = B^2 + D^2 - 1 = 0$ can be expressed, using substitutions A, B and D, as a function $R = f(L)$. Then,

$$M = B^2 + D^2 - 1 = \left(-\frac{L}{a}\right)^2 + \left(-\frac{a.R + b.L}{a.c}\right)^2 - 1 = 0 \quad (18)$$

By algebraic simplification, the equation (18) can be expressed as

$$aa.R^2 + bb.R + cc = 0 \tag{19}$$

where $aa = \frac{1}{c^2}$, $bb = -\frac{2.b.L}{a.c^2}$ and $cc = \frac{L^2}{a^2} + \frac{b^2.L^2}{a^2.c^2} - 1$.

Finally, the solutions of equation (19) are computed for all the possible variables L. Note that the variable L is restricted by the coordinate a of the sensor $\mathbf{S}_2$. Thus,

$$L \in \left\langle -a, a \right\rangle \text{ or } -\frac{a}{c_l} \leq \left(TOA_2 - TOA_1\right) \leq \frac{a}{c_l} \tag{20}$$

Figure 3 shows a curve covering all the points meeting the condition $M = 0$ in the hyperbolic plane.

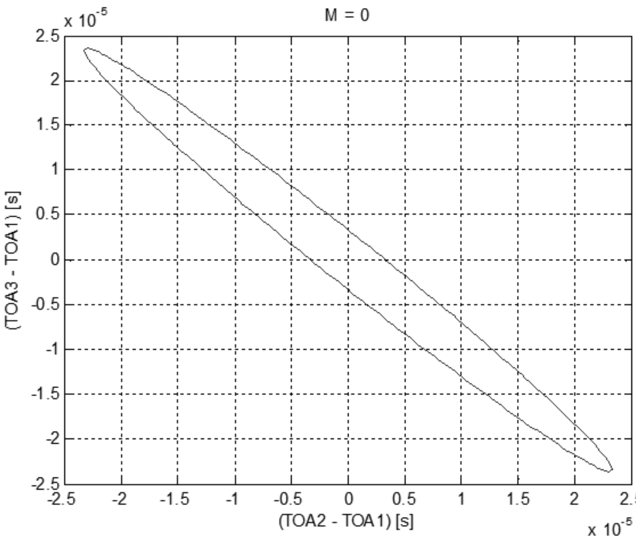

Figure 3. Representation of the condition $M = 0$ in the hyperbolic plane.

A representation of all points that satisfy the condition $M = 0$ directly in the plane $\{x, y\}$ isn't possible as there are points where the algorithm doesn't provide a solution. However, it is possible to use the following approach which is shown in Fig. 4.

This approach is based on an idea of finding all the areas in the coordinate system $\{x, y\}$ where both roots $K_{1,2}$ of Eqn. (15) are positive. It means the TDOA algorithm provides two "real" target positions. The areas marked Area 2 and Area 3 in Fig. 4 satisfy this condition. Strictly speaking, the first target lies in Area 2 and the second target in Area 3 or vice versa. Then, the border between these areas represents the $M = 0$ condition. In terms of physical interpretation, it has the following meaning. The targets that would lie at this border would be at an infinite distance from the sensor $\mathbf{S}_1$. For completeness, the following should be noted: Firstly, Area 1 shows the area where the TDOA equations have one "real" solution ($K > 0$) and one "non-real" solution ($K < 0$). Secondly, the border between Area 1 and Area 2 represents situations where Eqn. (15) has only one root, i.e. the determinant of (15) is equal to zero. It means the TDOA algorithm provides only one solution and it is "real".

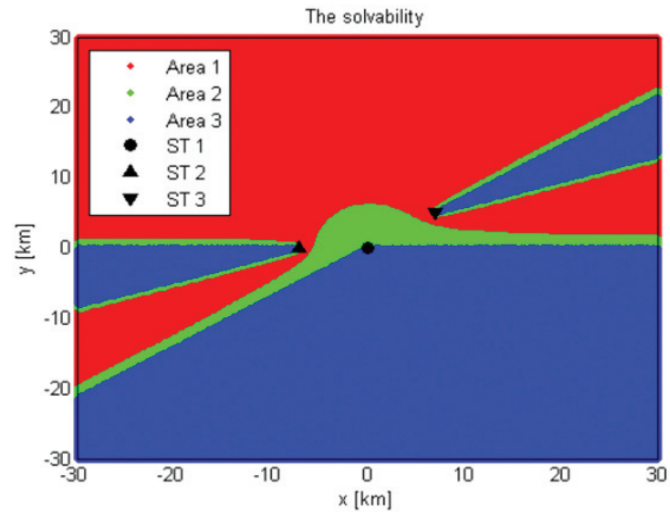

Figure 4. Representation of the condition $M = 0$ in the plane $\{x, y\}$.

3.2. TDOA Accuracy Analysis

In estimation theory, the Cramér-Rao Lower Bound (CRLB) expresses a lower bound on the variance of unbiased estimators of a deterministic parameter Eqn.(22). The approach which is based on CRLB theory is used for analyzing accuracy of the TDOA method. According to Eqn. (23), Eqn. (24) and Eqn. (25), it is possible to determine the covariance matrix $\mathbf{C}\left(\mathbf{T}\right)$ of the TDOA method as

$$\mathbf{CRLB}\left(\mathbf{T}\right) = \mathbf{C}\left(\mathbf{T}\right) = \mathbf{J}\left(\widehat{\mathbf{TOA}}\right)\mathbf{C}_p\left(\widehat{\mathbf{TOA}}\right)\mathbf{J}\left(\widehat{\mathbf{TOA}}\right)^T \tag{21}$$

where $\mathbf{J}\left(\widehat{\mathbf{TOA}}\right)$ is Jacobian Matrix (it consists of real partial derivatives of the function $f\left(\mathbf{TOA}, \mathbf{S}_{1..3}\right)$ with respect to the variables TOA_1 to TOA_3 for the measured vector $\widehat{\mathbf{TOA}}$)

$$\left(\widehat{\mathbf{TOA}}\right) = \left[\frac{\partial f\left(\widehat{\mathbf{TOA}}, \mathbf{S}_{1..3}\right)}{\partial \mathbf{TOA}}\right] = \begin{bmatrix} \frac{\partial x\left(\widehat{\mathbf{TOA}}, \mathbf{S}_{1..3}\right)}{\partial TOA_1} & \frac{\partial x\left(\widehat{\mathbf{TOA}}, \mathbf{S}_{1..3}\right)}{\partial TOA_2} \\ \frac{\partial y\left(\widehat{\mathbf{TOA}}, \mathbf{S}_{1..3}\right)}{\partial TOA_1} & \frac{\partial y\left(\widehat{\mathbf{TOA}}, \mathbf{S}_{1..3}\right)}{\partial TOA_2} \end{bmatrix} \tag{22}$$

and $\mathbf{C}_\mathrm{p}\left(\widehat{\mathbf{TOA}}\right)$ is the covariance matrix of the vector $\widehat{\mathbf{TOA}}$. If the time of arrivals on particular sensors are measured independently, as in the considered TDOA method, the matrix $\mathbf{C}_\mathrm{p}\left(\widehat{\mathbf{TOA}}\right)$ becomes a diagonal matrix in the following form:

$$\mathbf{C}_p\left(\widehat{\mathbf{TOA}}\right) = \begin{bmatrix} \sigma^2_{TOA1} & 0 & 0 \\ 0 & \sigma^2_{TOA2} & 0 \\ 0 & 0 & \sigma^2_{TOA3} \end{bmatrix} \tag{23}$$

An example of finding partial derivatives of the function $f\left(\mathbf{TOA}, \mathbf{S}_{1..3}\right)$ is shown in Appendix A.

Thus defined covariance matrix $\mathbf{C}\left(\mathbf{T}\right)$ represents a confidence region which includes "true" target position with a certain probability level[26]. From physical point of view the covariance matrix expresses an error ellipse with a given probability of "true" target occurrence[27]. The computation of parameters of the error ellipse, i.e. lengths of axes and their directions are detailed described in[28].

The following simulation is performed to verify the accuracy of the TDOA method. We assumed the same sensor arrangement as in the previous simulation. Next, we suppose the TOAs are measured at each sensor independently and all the sensors are time-synchronized. Standard deviation of the time of arrival measurement is equal to 10 ns for each sensor.

Then, the covariance matrix $\mathbf{C}\left(\mathbf{T}\right)$ is computed for all the possible target locations within area $x \in -50\text{km}, 50\text{km}$ and $y = -50\text{km}, 50\text{km}$ with a step of 500m. This creates an "accuracy map" of the TDOA method. The lengths of major axis of the corresponding error ellipses are shown in Fig. 5.

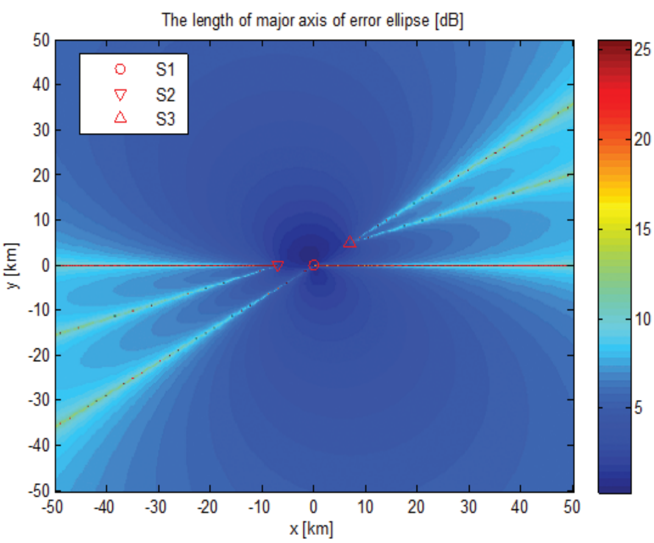

Figure 5. The "accuracy map" of the TDOA method

Note: The length of major axis of the error ellipse is used here due to high eccentricity of the error ellipses, and therefore, the CEP (Circular Error Probability) wasn't significant. Next, the logarithmic scale (to 0.1m) is used for better graphical interpretation of results.

Generally, this way of calculating the covariance matrix provides a detailed description of the TDOA method accuracy for both the arbitrary sensors network arrangement and the arbitrary target position in the area of interest. In addition, the Fig. 5 shows that there are places where the error of the method increases significantly. These areas are located around the lines intersecting the positions of sensors. In comparison with solvability analysis it can be stated that there are just the areas where the $M = 0$ condition is satisfied. This finding corresponds to the physical significance of the condition $M = 0$ (i.e. the method has no real solution).

4. ANALYTICAL SOLUTION OF TDOA TECHNIQUE FOR AN OVERDETERMINED SYSTEM

The assumed scenario for the target localization in this section is an overdetermined TDOA system, where the number of the measured TOAs is greater than the number of the unknown. In other words, it consists of one target and more than three known receiving sensors in TDOA 2D situation or more than four sensors in TDOA 3D situation. Overdetermined TDOA systems are quite often used in technical applications for some of their advantages, such as better coverage of the area of interest, elimination of solution ambiguity problem, etc. On the other hand, determination of a target position is more complicated in such systems. Usually, methods based on an iterative approach of solving nonlinear equations are used. Taylor-Series Estimation Method or Gauss (or Gauss-Newton) Interpolation Method is one of the best-known representatives of these iterative methods[29-30]. The performance of these methods is based on an iterative scheme to find a solution for a set of algebraic position equations (nonlinear equations), starting with a rough initial guess and improving the guess at each step by determining the local linear least sum squared error correction.

Assumed is a network composed of a set of *N* fixed positions and time-synchronized receiving sensors. There are sensors S_1 ... S_n, and one target T. The number of sensors $N>3$ represents an overdetermined system in 2D. This situation is shown in Fig. 6.

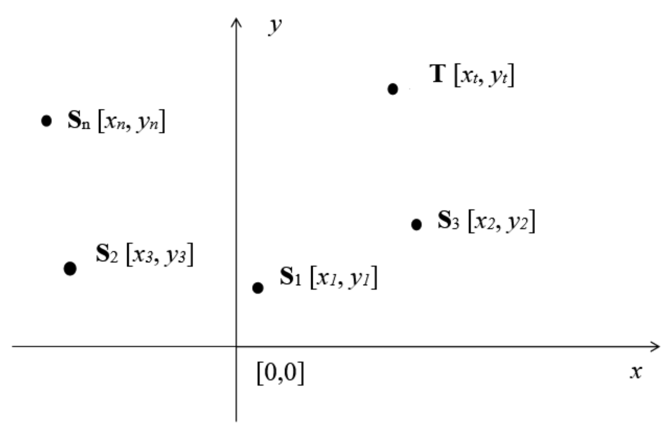

Figure 6. Arrangement of the overdetermined TDOA system.

Then, the position of the target can be found as follows. From the overall number of *N* sensors, $M = \binom{N}{3}$ combinations of sufficiently determined TDOA systems TS_1, .. ,TS_M can be created. Then, the target position T_i and its covariance matrix C_i can be calculated by TS_i TDOA system using algorithms mentioned in sections above. Finally, the target position T is calculates as[31]

$$\mathbf{T} = \left[\mathbf{T}_1.\mathbf{C}_1^{-1} + ... + \mathbf{T}_i.\mathbf{C}_i^{-1} + ... + \mathbf{T}_M.\mathbf{C}_M^{-1}\right].\left[\mathbf{C}_1^{-1} + ... + \mathbf{C}_i^{-1} + ... + \mathbf{C}_M^{-1}\right]^{-1} \quad (24)$$

where $\left[\;\right]^{-1}$ indicates inverse matrix.

Equation (24) represents a weighted arithmetic mean of particular target positions, where the weights are the corresponding covariance matrices. Figure 7 shows a graph of target position calculation in an overdetermined system. A sensor network composed of four receiving sensors with coordinates $\mathbf{S}_1$[0,0], $\mathbf{S}_2$[-7000m,7000], $\mathbf{S}_3$[7000m,7000m] and $\mathbf{S}_4$[0,-10000m] was assumed for this simulation. The target position was **T**[10000m,50000m]. All the TOAs were measured independently at all the sensors and they were expressed as a vector of TOAs estimations $\widehat{\mathbf{TOA}}$ (the vector $\mathbf{TOA}$ corresponds to the target position **T** that was burdened with a measurement error σ_{TOA} = 10ns). Then, the four different target positions $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ and $\mathbf{T}_4$ were provided by the TDOA algorithm. At the same time, the corresponding covariance matrices $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ and $\mathbf{C}_4$ were calculated. Finally, the position of the target $\mathbf{T}_e$ = [9984m, 49940m] using (27) was determined. The absolute difference between **T** and $\mathbf{T}_e$ was ΔR = 62.1m.

Note, an overall covariance matrix $\mathbf{C}_e$ of an overdetermined TDOA system can be calculated as

$$\boldsymbol{C}_e = k_e \cdot \left[\boldsymbol{C}_1^{-1} + \ldots + \boldsymbol{C}_i^{-1} + \ldots + \boldsymbol{C}_M^{-1} \right]^{-1} \tag{25}$$

where k_e is the correction factor.

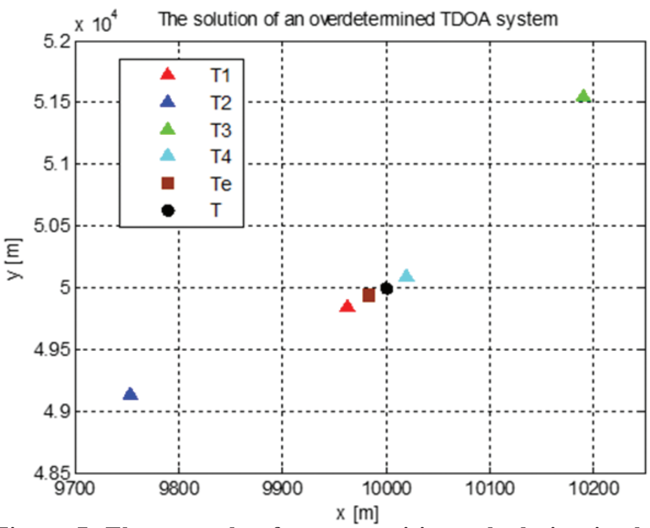

Figure 7. The example of target position calculation in the overdetermined TDOA system.

The correction factor k_e expresses the degree of independence of the system. In other words, it describes mutual independence of the individual subsystems in the overdetermined systems. The factor k_e is calculated from the following formula

$$k_e = \frac{M.r}{s} \tag{26}$$

where M is the number of combination of sufficiently determined systems, r is the number of mutually dependent sensors, and s is the minimum number of sensors of sufficiently determined systems.

In the example above, k_e was $k_e = \frac{M.r}{s} = \frac{4.2}{3} = 2.66$ and the overall covariance matrix was

$$\mathbf{C}_e = \begin{bmatrix} 4433 & 17936 \\ 17936 & 75890 \end{bmatrix}.$$

If all the individual subsystems are mutually independent, then $k_e = 1$.

5. BENEFITS OF THE TDOA ANALYTICAL SOLUTION IN PRACTICAL APPLICATIONS

The core of the presented analytical method allows to find a target position for both the overdetermined TDOA systems and for the sufficiently determined systems. The presented analytical solution of the TDOA technique also allows a clear evaluation of the accuracy of this method and the unambiguity of its solution. Thus, the main contribution of the presented paper is a complete analytical description of the TDOA method in 2D space (this limitation is only for the sake of clarity of the mathematical derivation), which is an essential condition for the practical application of this method. It is clear this method can be simply extended for 3D application which is very useful for all the TDOA systems. The proposed analytical solution of the TDOA localization technique (extended to 3D space) is computed many times per second in these 3D applications. The number of computation cycles depends on the number of targets, signal parameters (especially pulse repetition frequency), etc. Then, time optimization of the computational algorithm is important and for that a thorough analysis of the analytical solution of the TDOA technique is necessary. The proposed method, as well as its matrix form description, always (for all the measurements of the target positions) has the same number of elementary mathematical operations, and thus, the same computational time. This is the key condition for performing time optimization of this algorithm.

Analysis of the mathematical complexity and the computational speed of this algorithm in 3D space, based on a 3D extension of the proposed algorithm, and its matrix form description was already performed[32]. The results of this analysis showed that the analytical algorithm (proposed above) needed approximately 26% less mathematical operations than the matrix solution and that it eliminated problems of solving inverse matrices.

6. CONCLUSION

In summary, a complex description of a TDOA localization method was presented, including accuracy and solvability analysis. This approach for a comprehensive analytical description of the TDOA method was chosen due to its possible further use in practice, as already indicated in Section 5. An overall summary of all the benefits of this comprehensive TDOA localization technique approach is described below.

The Time Difference of Arrival localization technique is one of the very efficient transmitter localization methods used for long and short ranges. The principle of the method is based on a transformation from so-called Time Delay Space (i.e. Hyperbolic Layer in 2D or Hyperbolic Space in 3D application) to Cartesian Space in which the coordinates of the transmitter position are required. This transformation may be accomplished by using an iterative or an analytical method.

Both of these methods can perform this transformation even in a specific configuration of sensors, where their number corresponds to a spatial dimension (N+1) or to an overdetermined configuration.

This article was focused on an analytical method mainly for its advantages such as its constant speed, the ability to compute exact number of solutions including their values, the ability to estimate exact error, the possibility to derivate the target ambiguity and its pure geometrical representation of results. As mentioned above, there are three basic analytical solutions of the TDOA localization method – via trigonometric functions, via matrix solution and the presented method. The core of the presented method is free of trigonometric functions or inverse matrix solutions, and instead of several basic mathematical operations it needs only one square root operation. This was also the reason why the above presented method was the focus of this paper.

In contrast to the analytical methods, the main disadvantage of an iterative method is its dependency on the starting guess. An incorrect guess could lead to finding an incorrect target position due to the inherent ambiguity of the TDOA method. This is hard to handle especially at places close to borders of the discussed ambiguous areas. A second disadvantage is convertibility of the iteration method generally. However, these disadvantages of iteration TDOA methods are not connected to the overdetermined TDOA systems, and it may even be faster than analytical solutions based on a combination of non-overdetermined analytical algorithms and weighting. Of course, the main advantage of analytical solutions relates to the possibility to detect inaccurate TOA measuring and the ability to find an appropriate target position instead of cycling convergence with high error of iteration method.

The presented algorithm is one of the fastest methods due to its direct approach for both the stationary and the mobile receiver configurations and is able to visualize its solvability and ambiguity. It is also able to directly express an error estimation of the TDOA method, is free of numerous iterations, free of inverse matrix solution problems and free of trigonometric functions as shown above.

REFERENCES

1. Xiong, Hui; Chen, Zhiyuan; AN, Wei; Yang, Beiya. Robust TDOA Localization Algorithm for Asynchronous Wireless Sensor Networks. *Int. J. Distrib. Sens. Networks* [online]. 2015. ISSN 1550-1477.
   doi:10.1155/2015/598747.
2. Gu, Y.; Lo, A.; NIEMEGEERS, I. A survey of indoor positioning systems for wireless personal networks. *IEEE Commun. Surv. Tutor.,* 2009, 13–32.
   doi:10.1109/SURV2009.090103.
3. Raper, Jonathan; Gartner, Georg; Karimi, Hassan; Rizos, Chris. A critical evaluation of location based services and their potential. *J. Locat. Based Serv.* [online]. 2007, 5-45. ISSN 1748-9725.
   doi:10.1080/17489720701584069.
4. Khalaf-Allah, Mohamed. Time of arrival (TOA)-based direct location method. *In* 2015 16th International Radar Symposium (IRS) [online]. IEEE**,** 2015, 812-815. ISBN 978-3-9540-4853-3.
   doi:10.1109/IRS.2015.7226229.
5. Matousek, Zdenek; Babjak, Marian; Ochodnicky, Jan. Correlation-based TDOA algorithm for the target localization at low signal to noise ratio. In 2016 New Trends in Signal Process. (NTSP) [online]. IEEE, 2016, 1-5. ISBN 978-8-0804-0529-8.
   doi:10.1109/NTSP.2016.7747787.
6. Sidorenko, Juri; Schatz Volker; Bulatov, Dimitri; Scherer-Negenborn, Norbert, Arens, Michael & Hugentobler, Urs. Self-Calibration for the Time Difference of Arrival Positioning. *Sensors* [online]. 2020. ISSN 1424-8220.
   doi:10.3390/s20072079.
7. Fink, Andreas; Beikirch, Helmut. Analysis of RSS-based location estimation techniques in fading environments. *In* 2011 International Conference on Indoor Positioning and Indoor Navigation [online]. IEEE, 2011, 1-6. ISBN 978-1-4577-1805-2.
   doi:10.1109/IPIN.2011.6071943.
8. Li, Jinzhou; Guo, Fucheng; Yang, Le; Jiang, Wenli & Pang, Hongwei. On the use of calibration sensors in source localization using TDOA and FDOA measurements. *Digital Signal Process.* [online]. 2014, 33-43. ISSN 10512004.
   doi:10.1016/j.dsp.2013.12.011.
9. Almaaitah, A.; Alsaify, B. & Bani-Hani, R. Three-Dimensional Empirical AoA Localization Technique for Indoor Applications. *Sensors.* 2019.
   doi:10.3390/s19245544.
10. Nemati, M.; Baykas, T. & Choi, J. Performance of TDOA and AOA Localization Techniques for Different Base-Stations Topologies. *In* Proceedings of the 13th International Conference on Signal Processing and Communication Systems (ICSPCS); Institute of Electrical and Electronics Engineers, **2019,** pp. 1–7.
11. Kang, S.; Kim, T. & Chung, W. Hybrid RSS/AOA Localization using Approximated Weighted Least Square in Wireless Sensor Networks. *Sensors*, 2020. doi:10.3390/s20041159.
12. Foy, W. Position-Location Solutions by Taylor-Series Estimation. *IEEE Trans. Aerosp. Electron. Syst*. 1976, 187–194.
    doi:10.1109/TAES.1976.308294.
13. Kovavisaruch, L.; Ho, K.C. Modified Taylor-series Method for Source and Receiver Localization Using TDOA Measurements with Erroneous Receiver Positions. *In* 2005 IEEE International Symposium on Circuits and Systems. [online]. IEEE, 2005, 2295-2298. ISBN 0-7803-8834-8.
    doi:10.1109/ISCAS.2005.1465082.
14. Smith, J.; Abel, J. Closed-form least-squares source location estimation from range-difference measurements. *In* IEEE Transactions on Acoustics, Speech, and Signal

Process. [online]. 1987, 1661-1669. ISSN 0096-3518. doi:10.1109/TASSP.1987.1165089.

15. Wu, Peng; Su, Shaojing; Zuo, Zhen; Guo, Xiaojun; Sun, Bei & Wen, Xudong. Time Difference of Arrival (TDoA) Localization Combining Weighted Least Squares and Firefly Algorithm. *Sensors* [online]. 2019. ISSN 1424-8220.
doi:10.3390/s19112554.
16. Xiao, Jun; REN, Lirong & Tan, Jindong. Research of TDOA Based Self-localization Approach in Wireless Sensor Network. *In 2*006 IEEE/RSJ International Conference on Intelligent Robots and Systems. [online]. IEEE, 2006, 2035-2040. ISBN 1-4244-0258-1.
doi:10.1109/IROS.2006.282415.
17. Zhang, Yu; Gao, Kai & Zhu, Jiang. A TDOA-Based Three-Dimensional Positioning Method for IoT. *In* Proceedings of the 2018 International Conference on Mechanical, Electronic, Control and Automation Engineering. *(MECAE 2018)* [online]. 2018. ISBN 978-94-6252-493-4.
doi:10.2991/mecae-18.2018.136.
18. Bancroft, Stephen. An Algebraic Solution of the GPS Equations. *IEEE Trans. Aerosp. Electron. Sys.*[online]. 1985, 56-59. ISSN 0018-9251.
doi:10.1109/TAES.1985.310538.
19. Mellen, G.; Pachter, M. & Raquet, J. Closed-form solution for determining emitter location using time difference of arrival measurements. *IEEE Trans. Aerosp. Electron. Sys.* [online]. 2003, 1056-1058. ISSN 0018-9251.
doi:10.1109/TAES.2003.1238756.
20. Cui, Xunxue; Yu, Kegen & Lu, Songsheng. Approximate Closed-Form TDOA-Based Estimator for Acoustic Direction Finding via Constrained Optimization. *IEEE Sens. J.* [online]. 2018, 3360-3371. ISSN 1530-437X.
doi:10.1109/JSEN.2018.2803150.
21. Bard, J.D.; Ham, F.M. & Jones, W.L. An algebraic solution to the time difference of arrival equations. *In* Proceedings of SOUTHEASTCON '96 [online]. IEEE, 1996, 313-319. ISBN 0-7803-3088-9.
doi:10.1109/SECON.1996.510082.
22. Torrieri, Don. Statistical Theory of Passive Location Systems. *IEEE Trans. Aerosp. Electron. Syst.* [online]. 1984, 183-198. ISSN 0018-9251.
doi:10.1109/TAES.1984.310439.
23. Vesely, Jiri; Hubacek; Petr & Olivova Jana. The Power Gain Difference Method Analysis. *Sensors* [online]. 2020. ISSN 1424-8220, doi:10.3390/s20113018.
24. Liu, Zhixin; Wang, Rui & Zhao, Yongjun. A Bias Compensation Method for Distributed Moving Source Localization Using TDOA and FDOA with Sensor Location Errors. *Sensors* [online]. 2018. ISSN 1424-8220.
doi:10.3390/s18113747.
25. Li, Y.; Wang, C.; Qi, Q. & Sheng, A.. Closed-form formula of Cramer–Rao lower bound for 3D TOA target localisation. *Electron. Lett*. 2020, 43–45.
doi:10.1049/el.2019.2669.
26. Schervish, M.J. Theory of Statistics, Corr. 2nd printing 1996 Edition; Springer: New York, NY, USA, 1997. ISBN 978-0-387-94546-0.
27. Kay, S.M. Fundamentals of Statistical Signal Processing; Prentice Hall: Upper Saddle River, NJ, USA, 1993. ISBN 0133457117.
28. Lee, Joon-Ho; Cho, Sung-Woo & Choi, In-Sik. Simple expressions of CEP and covariance matrix for localization using LOB measurements for circular trajectory. *IEICE Electron. Express* [online]. 2012, 1221-1229. ISSN 1349-2543.
doi:10.1587/elex.9.1221.
29. Mantilla-Gaviria, Ivan; Leonardi, Mauro; Galati, Gaspare & Balbastre-Tejedor, Juan. Localization algorithms for multilateration (MLAT) systems in airport surface surveillance. *Signal, Image and Video Process.*[online]. 2015, 1549-1558. ISSN 1863-1703.
doi:10.1007/s11760-013-0608-1.
30. Romero, L. A. & Mason, Jeff. Evaluation of Direct and Iterative Methods for Overdetermined Systems of TOA Geolocation Equations. *IEEE Trans. Aerosp. Electron. Syst.* [online]. 2011, 1213-1229. ISSN 0018-9251.
doi:10.1109/TAES.2011.5751253.
31. Leonardo, Rodrigo & Santos, Marcelo; Cove, Karen. An Empirical Approach for the Estimation of GPS Covariance Matrix of Observations. *In* Proceedings of the 61st annual meeting of the Institute of Navigation (2005). 2005, 1098-1104.
32. Van Doan, Sang; VESELY, Jiri. The effectivity comparison of TDOA analytical solution methods. *In* 2015 16th International Radar Symposium (IRS). [online]. 2015, 800-805. ISBN 978-3-9540-4853-3. doi:10.1109/IRS.2015.7226350.

ACKNOWLEDGEMENT

The work presented in this article has been supported by the Czech Republic Ministry of Defence – University of Defence development program "Airspace Operations".

APPENDIX A

The Jacobian matrix **J** is defined as:

$$\mathbf{J}\left(\widehat{\mathbf{TOA}}\right)=\left[\frac{\partial f\left(\widehat{\mathbf{TOA}},\mathbf{S}_{1..3}\right)}{\partial\mathbf{TOA}}\right]=\begin{bmatrix}\frac{\partial x\left(\widehat{\mathbf{TOA}},\mathbf{S}_{1..3}\right)}{\partial TOA_1} & \frac{\partial x\left(\widehat{\mathbf{TOA}},\mathbf{S}_{1..3}\right)}{\partial TOA_2} & \frac{\partial x\left(\widehat{\mathbf{TOA}},\mathbf{S}_{1..3}\right)}{\partial TOA_3}\\ \frac{\partial y\left(\widehat{\mathbf{TOA}},\mathbf{S}_{1..3}\right)}{\partial TOA_1} & \frac{\partial y\left(\widehat{\mathbf{TOA}},\mathbf{S}_{1..3}\right)}{\partial TOA_2} & \frac{\partial y\left(\widehat{\mathbf{TOA}},\mathbf{S}_{1..3}\right)}{\partial TOA_3}\end{bmatrix} \quad \text{(A-1)}$$

An example of a calculation of partial derivatives is described below. For example, the x_t target coordinate is obtained by equation

$$x_t = A + B.K \qquad \text{(A-2)}$$

where $A = \dfrac{a - L^2}{2.a}$ and $B = \dfrac{-L}{a}$ are substitutes.

Then, the partial derivate of x_t with respect to TOA_1 is

$$\frac{\partial x\left(\widehat{\mathbf{TOA}},\mathbf{S}_{1..3}\right)}{\partial TOA_1} = \frac{\partial A}{\partial TOA_1} + \frac{\partial B}{\partial TOA_1} \cdot K + B \cdot \frac{\partial K}{\partial TOA_1} \qquad \text{(A-3)}$$

here the partial derivatives of substitutes A and B with respect to TOA_1 are

$$\frac{\partial A}{\partial TOA_1} = -\frac{L}{a}.\frac{\partial L}{\partial TOA_1} = \left| \begin{matrix} \text{if } L = \left(TOA_2 - TOA_1\right).c_l \\ \text{then} \dfrac{\partial L}{\partial TOA_1} = -c_l \end{matrix} \right| = \frac{L}{a}.c_l \qquad \text{(A-4)}$$

$$\frac{\partial B}{\partial TOA_1} = -\frac{1}{a}.\frac{\partial L}{\partial TOA_1} = \frac{1}{a}.c_l \qquad \text{(A-5)}$$

If K is expressed as

$$K = \frac{-N \pm \sqrt{N^2 - 4.M.P}}{2.M} \qquad \text{(A-6)}$$

where $N = A.B + C.D$, $M = B^2 + D^2 - 1$ and $P = A^2 + C^2$. Then, the partial derivative of K with respect to TOA_1 is

$$\partial K \qquad 2.\left[\frac{\partial\left(-N \pm \sqrt{N^2 - 4.M.P}\right)}{\partial TOA_1}\right].M - 2.\left[-N \pm \sqrt{N^2}\right. \cdot \qquad \text{(A-7)}$$

The derivatives $\left[\dfrac{\partial\left(-N \pm \sqrt{N^2 - 4.M.P}\right)}{\partial TOA_1}\right]$ and $\dfrac{\partial M}{\partial TOA_1}$

must be solved taking into account the fact that variables (or substitutes) C, D, N, M and P are depended on TOA_1 too. That means that it is necessary to calculate their partial derivatives with respect to TOA_1 too. The remaining partial derivatives of the Jacobian matrix can be derived in the same way.

CONTRIBUTORS

Dr Petr Hubacek graduated as PhD on 'The Optimization of TDOA System Topology in terms of Target Location Accuracy' in 2010 at University of Defense in Brno, Czech Republic. Main field of his study is aimed to the technology of EW systems and modern signal source location techniques and algorithms. After serving for 5 years at EW unit of the Czech Army he started his teacher's carrier at Radar Department at Military Academy in Brno. From 2010 he is head of Group of Electronic Warfare system of the Department of Communication Technology, Electronic Warfare and Radar Technology.
For this peper, he provided the error analysis of the method: investigation, data curation, writing—original draft preparation.

Assoc Prof Jiri Vesely graduated as PhD on Low Flying Target location using Surface Seismic Waves in 2001, and as Assoc Prof with Habilitation treatise on modern ELINT system principles extension at University of Defense in Brno, Czech Republic in 2012. Main field of study is modern signal source location principles and algorithms, radar signal analysis and classification for ELINT and EW systems, radar tracking and data fusion in complicated environment. He began his teacher carrier in 1996 at Radar Department in Military Academy in Brno and from 2019 he is leader of Department of Communication Radar and Electronic Warfare Technology, Faculty of Military Technology, University of Defense.
For this paper, he put forward the TDOA method algorithm: conceptualization, methodology, formal analysis, writing—review and editing.

Dr Jana Olivova graduated as PhD on Multiobjective Optimization in EMC in 2011 at Brno University of Technology, Czech Republic. The work was aimed to propose a methodology for creating an equivalent of composite materials used for construction of small aircraft. In order to the equivalent of composite materials, global optimization methods was used. Since 2011, she has been working at the University of Defense as a senior scientist and she is dedicated to the designs of antenna systems and the optimization of the TDOA method.
For this paper, she prepared all simulations of the TDOA localization technique: validation, writing—review and editing, project administration.