Defence Science Journal, Vol. 58, No. 2, March 2008, pp. 209-219

Multiscale Coupled Model of Thermoelastoviscoplasticity and Damage and its Application for Modelling Metal Forming Processes

V.N. Kukudzhanov

Russian Academy of Sciences, 119526, Vernadskogo av.101, b.1, Moscow, Russia

ABSTRACT

The micromechanical model of damage and fracture of elastovisoplastic materials is suggested and applied to studying the processes of damage and fracture. The model describes the scale effect in processes of damage and localization of plastic strain in adiabatic shear bands. For the integration of the suggested equations the new splitting method was used. It was shown that it has significant advantages compare to existing iterative methods. The possibilities of the model suggested for the description of the plastic strain localization are illustrated by several examples.

Keywords: Thermoelastoviscoplasticity, dislocations, microstress, constitutive equations

1. INTRODUCTION

The micromechanical model of damage and fracture of elastovisoplastic materials is suggested and applied to study the processes of metal forming and localisation of plastic strain. The model is based on the phenomenological Taylor-Gillman theory[1] of dislocations and microdefect development. According to this model, the plastic deformation and fracture is a single process caused by the motion of dislocation and also (at a later stage) by generation and development of microdefects. The deformation process is assumed to consist of two stages. At the first stage the material is subjected to plastic deformation due to the motion of dislocations in crystal grains of the material. Part of these dislocations is accumulated at intergrain boundaries. This concentration of dislocations leads to residual microstresses.

On the macrolevel, it leads to an elastoviscoplastic flow with simultaneous hardening of the material. The second stage begins after the intensity of the dislocation flux accumulated at the boundaries of grains attains a critical level, beyond which the process of annihilation of dislocations begins. This process is accompanied by disclinations of grains with the formation of voids between them. On the macrolevel, this process leads to relaxation of the internal stresses and softening of the material[2].

The flux of annihilating dislocations is characterized by the tensor of annihilation of dislocations. On the macrolevel, this flux leads to damage of the material and is described in terms of the damage tensor. The spherical part of the damage tensor is associated with the volume of appearing voids, while the deviatoric part is associated with the fracture shear deformation of the material leading to the relaxation of the residual stresses. We assume that the voids to be nearly spherical. The effective elastoviscoplastic material with spherical voids can be described by the associated law of plastic flow depending on the content of voids in the material and on the plastic strain rate. A complete system of constitutive equations for macroparameters relates the tensors of effective

Received 30 October 2007

and residual stresses with the tensors of viscoplastic strain rate and damage.

The obtained system of equations consist three different time constants corresponding to characteristic size of considered body and two relaxation times τ_p (porous relaxation time) and τ_d (dislocation relaxation time). In dimensionless form the equations system has two small parameters $\delta_d = \tau_d/t_0$ and $\delta_p = \tau_p/t_0$. At the first hardening stage, the scale effect is determined by parameter δ_d and on the second softening (damage) stage by parameter[3] δ_p ? δ_d. When $\delta_p \to 0$ and $\delta_d \to 0$, the model reduces to GTN-model.

The investigation of the system shows that it does not contradict the thermodynamic inequalities and provides well-posed formulations of boundary value problems of mechanics, including those governing the stages of softening of the material. This permits one to use the model in problems for describing the process of strain localisation and damage of material.

The possibilites of the model suggested for the description of the plastic strain localisation are illustrated by several examples.

2. DIFFERENT STAGES OF SOFTENING OF MATERIAL

2.1 First Stage

A model of an thermoelastoviscoplastic medium coupled with the damage has been suggested which causes the formation of microdefects that lead to gradual fracture of the material and intensive plastic flow in localized regions resembling slip bands is suggested.

On the microlevel, the initial stage of the plastic flow is described by the motion of dislocations. The rate of the viscoplastic strain, $\dot{\gamma}^p$, is proportional to the flux of mobile dislocations, i.e.,

$$\dot{\gamma}^p = abN_m V \tag{1}$$

where a is the Burgers vector and b is the orientation coefficient. The average speed of dislocations, V, determined by the motion due to thermal fluctuations and the acting effective stress $\sigma^a = \sigma - \sigma^r$ is given by

$$V = V_0 \exp\left(\frac{U_0 - (\sigma - \sigma^r)}{-k\Theta}\right), \qquad s \geq s^r \tag{2}$$

where U_0 is the activation energy, k, the Boltzmann constant, Θ the absolute temperature, and σ^r, the residual stress.

The number of mobile dislocations N_m increases with the plastic strain γ_P according to a power law and decreases with the total number of dislocations, N exponentially, due to the effect of locking of dislocations at the intergrain boundaries[4,5]. One has

$$N_m = (N_0 + a\gamma_p)^n \exp\left(-\frac{N}{N^*}\right) \tag{3}$$

where N_0, N^*, and a are material constants.

To describe the generation and development of microdefects, it is necessary to take into account the balance of dislocation fluxes in the material. In accordance with Eqns (1)–(3), the total dislocation flux $\dot{p}_{ij}$ at the first stage of the plastic deformation is divided into two components–the flux of mobile dislocations $\dot{\gamma}^p_{ij}$, which in fact forms the plastic deformation, and the flux of dislocations $\dot{\omega}_{ij}$ condensing at isolated inclusions and the grain boundaries.

Let η be the fraction of the flux $\dot{p}_{ij}$ associated with mobile dislocations and let $1-\eta$ be the fraction of the flux condensing at the grain boundaries. Then one can write

$$\dot{\gamma}^p_{ij} = \eta \dot{p}_{ij}, \quad (1-\eta)\dot{p}_{ij} = \dot{\omega}_{ij}, 0 < \eta < 1 \tag{4}$$

which implies

$$\dot{\omega}_{ij} = (1-\eta)\dot{\gamma}^p_{ij}/\eta \, .$$

When the degree of condensation of dislocations at the intergrain boundaries reaches a critical value, partial annihilation of the dislocations takes place, the grains begin to move relative to one another, disclinations appear and microvoids and microcracks are formed.

This second stage of the deformation is characterised by gradual fracture of the material, which leads to an additional deformation. The plastic strain is concentrated in the regions of the most intensive microdamage and leads to the development of shear bands. The equations of balance of fluxes at the second stage has the form

$$(1-\eta)\dot{p}_{ij} = \dot{\omega}_{ij} + \dot{b}_{ij}, \tag{5}$$

where b_{ij} is the dislocation annihilation flux tensor. Note that at the first stage of the plastic deformation the spherical part of this tensor is equal to zero, i.e., $p_{ii} = 0$, and hence $\omega_{ii} = 0$, while at the second stage $p_{ii} \neq 0$ and $b_{ii} \neq 0$.

It is natural to assume that the flux of annihilation of dislocations, δ_τ, is proportional to the amount of dislocations accumulated at the inclusions, $\dot{\omega}_{ij}$, and that these quantities are related by

$$\dot{b}_{ij} = \lambda\omega_{ij}, \tag{6}$$

where λ is an unknown scalar multiplier.

The damage process begins only after the intensity of the tensor of dislocations accumulated at the inclusions, $\Omega = \frac{1}{2}\sqrt{\omega_{ij}\omega_{ij}}$, attains a critical value Ω_0. In this case, the annihilation intensity, $\dot{B}_{\text{II}}$, is a monotone function of the excessive intensity $\Omega_{\text{II}} - \Omega_0$,

$$\dot{B}_{\text{II}} = \left(\frac{1}{2}\dot{b}_{ij}\dot{b}_{ij}\right)^{1/2} = \frac{1}{\tau_p}\hat{Q}(\Omega_{\text{II}} - \Omega_0),$$
$$\hat{Q}(z) = \begin{cases} Q(z) & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases} \tag{7}$$

where $Q(z)$ is a dimensionless function and τ_p is a time of relaxation of ω_{ij} or residual stress s^r_j measured in seconds $(\tau_p ? \tau_d)$. Using Eqn (7), one can determine the parameter λ

$$\lambda = \frac{\hat{Q}(\Omega_{\text{II}} - \Omega_0)}{\tau_p\Omega_{\text{II}}} \tag{8}$$

Based on Eqns (4)–(8), one can write the final equation for the flux ω_{ij} in the form

$$\frac{d\omega_{ij}}{dt} + \frac{\hat{Q}(\Omega_{\text{II}} - \Omega_0)}{\tau_p\Omega_{\text{II}}}\omega_{ij} = \frac{1-\eta}{\eta}\frac{d\gamma^{\text{p}}_{ij}}{dt} \tag{9}$$

The critical density of dislocations, Ω_0, at which microcracks appear on the intergrain boundaries and other obstacles, and the corresponding critical stress intensity S_0 can be determined on the microlevel and was calculated by a number of authors[6,7] who suggested various kinds of microfracture models.

In macrophysical terms, the tensor ω_{ij} corresponds to the microstress tensor. This follows from the familiar experimental fact that the microstress magnitude is proportional to the density of dislocations accumulated at the grain boundaries[4]. Accordingly, the condition $\Omega_{\text{II}} = \Omega_0$ can be interpreted as the macro equation of the surface in the space of the stress tensor components corresponding to the beginning of the fracture.

To proceed from the microparameters to macroparameters, Eqns (1)–(3) will present for the case of the 3D stress-strain state in the generalised form relating the second invariant of the plastic strain rate tensor $\dot{\gamma}^{\text{p}}$ and the tensor of active stresses S^a_{II}

$$\dot{\gamma}^p = \frac{1}{\tau_d} f(\gamma^p)\Psi(S^a_{\text{II}} - T_s), \quad S^a_{\text{II}} = (\frac{1}{2}\sigma^a_{ij}\sigma^a_{ij})^{1/2},$$
$$\dot{\gamma}^p = (\frac{1}{2}\dot{\gamma}^p_{ij}\dot{\gamma}^p_{ij})^{1/2} \tag{10}$$

Using the hypotheses of the flow theory, one obtains from Eqns (1)–(3) the elastoviscoplastic equations of the form

$$\dot{\gamma}_{ij} = \frac{1}{2\beta}\dot{s}_{ij} + f(\gamma^p)\frac{\Psi(S^a_{\text{II}} - T_s)}{\tau_d S^a_{\text{II}}}\sigma^a_{ij},$$
$$\sigma_{ii} = 3K\varepsilon_{ii}, \quad \gamma_{ij} = \varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk} \tag{11}$$

For simplicity, one can assume the elastic law to be valid for the spherical part σ_{ii} of the stress tensor. This implies that the material is plastically incompressible, i.e., $\varepsilon^p_{ii} = 0$.

It is assumed that the deviatoric components of the residual stress tensor and the tensor of accumulated flux of dislocations are related by

$$s^r_{ij} = 2\mu^*\omega_{ij}' \tag{12}$$

then the condition $\Omega_{II} \leq \Omega_0$ corresponds to the condition $S^r_{II} \leq S_0$ for the residual stresses s^r_{ij}. The evolution equation for s^r_{ij} at the first stage of the plastic deformation takes the form

$$\frac{1}{2\mu^*}\dot{s}^r_{ij} = \frac{1-\eta}{\eta}\dot{\gamma}^p_{ij} \tag{13}$$

The integration of this equation with respect to t at constant μ^* yields the well-known law of kinematic hardening,

$$s^r_{ij} = 2\alpha\gamma^p_{ij}, \qquad \alpha = \mu^*\frac{1-\eta}{\eta} \tag{14}$$

The constant coefficient α can be determined on the basis of the experimental data taking into account the Bauschinger effect.

Thus, for $S^r_{II} < S_0$, i.e., for the period prior to the beginning of the formation of microvoids, the behaviour of the matrix material is described in accordance with the dislocation theory by Eqns (11)–(14) corresponding to a viscoplastic medium with kinematic hardening.

For the case of $S^r_{II} < S_0$, the relaxation equation for the tensor s^r_{ij} in the following form is obtained:

$$\dot{s}^r_{ij} + \frac{2\mu^*}{\tau_p}\frac{Q(S^r_{II} - S^r_0)}{S^r_{II}}s^r_{ij} = 2\alpha\dot{\gamma}^p_{ij} \tag{15}$$

where $\dot{\gamma}^p_{ij}$ is the deviatoric part of the plastic strain rate tensor.

Using the function $\hat{Q}(z)$ of Eqn (7), one can combine Eqns (13)–(15) to a single equation. Equation (15) describes the relaxation of the residual stresses in the material beyond the time instant at which the formation of the microvoids and, hence, the softening begins. Before this time the hardening occurred in the material in accordance with Eqn (14). From Eqn (15) it follows that during the relaxation stage the residual stress is reduced to a certain nonzero value S^r_0. For t ? τ_p, the residual stresses change in accordance with the law of an ideal plastic flow associated with the yield surface $S^r_{II} = S_0$ The relaxation equation for this case can be obtained from Eqn (15) by letting $\tau_p \to 0$, one has

$$ds^r_{ij} + H(s^r_{mn}d\varepsilon^r_{mn})\frac{s^r_{mn}d\varepsilon^r_{mn}}{(S^r_{II})^2}s^r_{ij} = 2\alpha d\gamma^p_{ij} \tag{16}$$

where $H(z)$ is the Heaviside unit step function.

2.1 Second Stage

The appearance of voids changes the behaviour of the material essentially. In this case, the material consists of the matrix and voids, i.e., it is a two-phase material. Knowing the matrix properties, which are described by Eqn (11), one can determine the effective characteristics of the plastic material. First of all, the appearance of voids leads to a change in the yield conditions of the material. Apart from this, equations describing the generation and development of defects are required. The difficulty of the solution of this problem substantially depends on of the shape of the voids. For nonlinear media, one can obtain a numerical solution and try to approximate this solution by a simple analytical expression taking into account most essential characteristics of the process or using an empirical expression based on the experimental data. First the simplest case of spherical voids is considered.

Gurson[8] solved this problem for an ideal plastic medium containing a spherical void and loaded at infinity by a biaxial nonuniform pressure. The yield condition for a porous medium was obtained in the form

$$F(\sigma_{ij}, f, \sigma_s) = \frac{3s_{ij}s_{ij}}{2\sigma_s^2} + 2fq_1\cosh\left(\frac{3q_2}{2}\frac{\sigma_{kk}}{\sigma_s}\right) - (1 + q_1^2 f^2) = 0 \tag{17}$$

where σ_{ij} is the stress in the porous material, σ_s the yield limit of the porous material, f the porosity, and q_1 and q_2 are adjusting parameters. One can generalise this condition to the porous material the matrix of which is elastoviscoplastic with kinematic hardening and is described by Eqns (10)–(11).

To this end, replace Mises yield condition [Eqn (17)] by the relation

$$F(t_{ij}, f, T_s) = \frac{3s_{ij}^a s_{ij}^a}{2T_s^2} + 2fq_1\cosh\left(\frac{3q_2}{2}\frac{\sigma_{kk}}{T_s}\right) - (1+q_1^2 f^2) = 0 \quad (18)$$

where s_{ij}^a is the deviatoric part of the active stress tensor and T_s is the yield limit of the porous elastoviscoplastic material. The quantity T_s is determined by the condition of equality of the plastic works for the matrix and the effective material. This condition leads to the equation

$$\begin{aligned} &\sigma_{ij}^a \dot{\varepsilon}_{ij}^p = (1-f)\dot{\gamma}^p[T_s(\gamma^p) + \Psi^{-1}(\tau_d \dot{\gamma}^p)] \\ &T_s = T_s(\gamma^p) + \Psi^{-1}(\tau_d \dot{\gamma}^p) \end{aligned} \quad (19)$$

The quantity γ^p is determined from the first equation and then the second equation is used to find T_S. The stress σ_{ij}^a and dissipative strain rate $\dot{\varepsilon}_{ij}^p$ in the effective material are related by the associated law of plastic flow

$$\dot{\varepsilon}_{ij}^p = \Lambda \frac{\partial F}{\partial \sigma_{ij}^a} \quad (20)$$

provided that the associated law is valid for the matrix of the material. The parameter Λ is determined from the second equation of Eqn (19) and has the form

$$\Lambda = \frac{T_s}{\tau_d}\Psi(T_s - T_s(\gamma^p))\left(\frac{\partial F}{\partial \sigma_{ij}^a}\sigma_{ij}^a\right)^{-1} \quad (21)$$

The characteristics of the annihilation tensor b_{ij} are associated with the damage of the medium. It was shown previously that the deviatoric part of this tensor, b_{ij}', was related to the fracture strain deviator and led to the relaxation of the residual stresses. The spherical part $\dot{b}_{ii}$ is proportional to the volume fracture strain $\dot{b}_{ii} = \dot{\varepsilon}_{ii}^{\mathrm{R}}$ and is related to the porosity f.

The continuity relation implies the equation for the porosity $f = \Delta V_{voids}/V$

$$\dot{f} = (1-f)\dot{\varepsilon}_{kk}^p = \Lambda \frac{3f(1-f)}{T_s} q_1 q_2 \sinh\frac{3q_2\sigma_{kk}}{2T_s} \quad (22)$$

2.2.1 Evolution of the Plastic Strain Intensity and Porosity

The hardening-softening of the matrix material is described by the dependence $\sigma_Y(\bar{\varepsilon}_m^{pl})$. Starting from the fact that the work of plastic strains is performed only by the matrix material, one obtains the following equation for the evolution of $\bar{\varepsilon}_m^{pl}$:

$$\begin{aligned} &(1-f)\sigma_Y \dot{\bar{\varepsilon}}_m^{pl} = \sigma : \dot{\varepsilon}_{pl} \\ &\bar{\varepsilon}_m^{pl} = \sqrt{\frac{2}{3}\varepsilon_m^{pl} : \varepsilon_m^{pl}} \end{aligned} \quad (23)$$

where ε_m^{pl} is the plastic strain of the matrix material. The material porosity f_{varies} owing to the growth of existing voids $\dot{f}_{gr}$ and the nucleation f_{nucl} of new ones. From the continuity equation, assuming that the matrix material is plastically incompressible, the equation for the void growth is derived

$$\dot{f} = \dot{f}_{nucl} + \dot{f}_{gr}., \quad \dot{f}_{gr} = (1-f)\dot{\varepsilon}^{pl} : \mathrm{I} \quad (24)$$

The void nucleation is due to the relative motion of grains and depends on the plastic strain intensity[9]

$$\begin{aligned} &\dot{f}_{nucl} = A\dot{\bar{\varepsilon}}_m^{pl} \\ &A(\bar{\varepsilon}_m^{pl}) = \frac{f_N}{s_N\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\bar{\varepsilon}_m^{pl} - \varepsilon_N}{s_N}\right)\right] \end{aligned} \quad (25)$$

The strain intensity at which voids start to nucleate obeys the normal distribution with mean ε_N and variance s_N. The volume fraction of the nucleating voids is equal to f_N [9]. The voids originate only in tension.

This equation completes the system of constitutive relations (18)–(20). Using Eqns (18)–(20), one can determine the stresses and internal parameters of the material for a given velocity field, which can be determined from the equations of motion.

For the model suggested, the fracture criterion of the material is the condition of the critical accumulation of voids. Since the grains of crystallites are constrained and cannot be deformed freely, intercrystallite voids are formed at the interfaces between the crystallites. These voids are accumulated to a critical value of the porosity at which the voids begin to spread catastrophically. This leads to a compete intercrystallite fracture of the material. This critical value of the porosity, f_*, depends on a number of external influences, in particular, on the temperature and the rate of loading. It also depends on the material structure and, according to the experiments[4], ranges between 0.05 and 0.5. It should be noted that the formation of voids is frequently associated with the formation of lattice vacancies and their motion toward the boundaries of the crystal at which the vacancies coagulate and voids are formed[4,7].

The main difference between the high temperature and low temperature fracture of semibrittle polycrystalline materials is in the fracture mechanism. In the former case, the defects have a large mobility due to high thermal activation, which provides these with an ability to produce a complete intercrystallite fracture. In the later case, the defects do not have enough time to decrease the stress concentration caused by a crack and prevent a catastrophic growth of this crack. This leads to a brittle fracture which is mostly accounted for by the propagation of isolated macrocracks[7].

One can readily take into account temperature effects by adding the thermal component $\varepsilon_{ij}^{\Theta} = \alpha\Theta\delta_{ij}$ to the strain ε. The temperature Θ should be found from the equation

$$\rho_e c_p \frac{\partial\Theta}{\partial t} = \chi \sigma_{ij}^{a} \dot{\varepsilon}_{ij}^{p} \tag{26}$$

which follows from the hypothesis of the adiabatic nature of the process. Here c_p is the heat capacity at constant stress, χ the thermal conversion coefficient, ranging between 0.8 and 0.9, α the thermal expansion coefficient, and ρ_e the density of the effective material.

If the voids are chaotically distributed over the material and do not have a dominating orientation in the loaded material, then the dominant influence on the change in the stress-strain state is that of the first invariant of the tensor ρ_{ij}. In the first approximation, this invariant satisfies Eqn (22), and hence, the model suggested can be utilised. The application of the model suggested can be illustrated by a number of specific examples.

Suggested model is transform to GTN model if the matrix material is strain rate independent $\delta_d \to 0$, $\delta_p \to 0$ and has isotropic hardening. The damage begins simultaneously with appearance of plasticity and in Eqn (18) $s_{ij}^{a} = s_{ij}$ and $T_s = \sigma_Y(\bar{\varepsilon}_m^{pl})$ is the yield limit of the the matrix material depending on the plastic strain intensity. The yield condition takes the form

$$\Phi = \left(\frac{q^2}{\sigma_Y}\right) + 2q_1 f \cosh\left(-\frac{3}{2}\frac{q_2 p}{\sigma_Y}\right) - (1 + q_1^2 f^2) = 0$$

$$s = pI + \sigma, \qquad q = \sqrt{\frac{3}{2} s : s}, \tag{27}$$

$$p = -\frac{1}{3}\sigma : I, \qquad f = \frac{V_{por}}{V}$$

Here s is the deviator of the Cauchy stress tensor, q is the shear stress intensity, p is the hydrostatic pressure, and f is the porosity (the volume fraction of voids in the material).

Tvergaard[10] introduced constants q_1 and q_2 into this condition (as coefficients correcting the influence of the porosity and pressure). By varying parameters q_i, one can make the results of numerical simulations closer to experimental data.

2.2.2 Associated Flow Law

The yield condition is taken as plastic potential and represent the plastic strain as

$$\dot{\varepsilon}^{pl} = \dot{\lambda}\frac{d\Phi}{d\sigma} = \dot{\lambda}\left(-\frac{1}{3}\frac{\partial\Phi}{\partial p}I + \frac{3}{2q}\frac{\partial\Phi}{\partial q}\mathrm{s}\right) \tag{28}$$

where $\dot{\lambda}$ is a nonnegative scalar factor. $\dot{f}$ are determind by Eqns (23)–(25).

3. SPLITTING METHOD

A new and effective numerical-analytical method for integrating the equations of elastoviscoplastic media with structural variables based on the splitting the constitutive relations with respect to physical processes was proposed by Kukudzhanov[11].

First, consider the integration of the constitutive equations of elastoplastic medium involving structural variables and independent of time scale variations.

The general scheme for splitting the elastoplastic equations can be described as follows[11]. Only the constitutive equations are split writing in the form

$$\frac{d\sigma}{dt} = D:(\dot{\varepsilon} - \dot{\varepsilon}^{pl}) \tag{29}$$

where D is the tensor of elastic moduli of the material and $d\sigma/dt$ is the objective derivative of the Cauchy stress tensor.

The predictor is taken for $\dot{\varepsilon}^{pl} = 0$; then at the step Δt, it is necessary to solve an elastic problem with the initial conditions obtained at the preceding step for the complete elastoplastic problem:

$$\frac{d\sigma}{dt} = D:\dot{\varepsilon} \tag{30}$$

After this, at the corrector stage, the stress relaxation Eqn (29) are solved for $\dot{\varepsilon} = 0$ together with the equations describing the internal structure of the material (hardening, damage, etc.) under the initial conditions obtained at the predictor stage.

The associated flow law (28) is used to obtain the stress relaxation equations

$$\frac{d\sigma}{dt} = -\frac{d\lambda}{dt}D\frac{\partial\Phi}{\partial\sigma}, \quad \frac{dH_i}{dt} = \frac{d\lambda}{dt}F_i(\sigma, H_j) \tag{31}$$

where H_i are parameters describing the internal structure of the material.

The problem of integrating the constitutive equations for stresses and internal parameters of the medium under given strains is a stand-alone problem for both the weak or the differential form of the conservation laws[11].

In the case of classical or equilibrium (steady state) elastoplastic medium whose properties are independent of variations in the time scale and it is possible to eliminate time t from Eqn (31) and pass to differentiation with respect to the variable λ:

$$\frac{d\sigma}{d\lambda} = -D\frac{\partial\Phi}{\partial\sigma}, \quad \frac{\partial H_i}{\partial\lambda} = F_i(\sigma, H_j) \tag{32}$$

Solving Eqn (32) with the initial conditions $\sigma(\lambda_0) = \sigma^{el}$ and $H_i(\lambda_0) = H_i^{el}$ obtained by solving the elastic problem, the solution is found as a function of the parameters λ, σ^{el}, and H_i^{el}

$$\sigma = \sigma(\lambda, \sigma^{el}, H_i^{el}), \quad H_i = H_i(\lambda, \sigma^{el}, H_i^{el}) \tag{33}$$

Substituting the obtained solution into the plasticity condition (28), one obtains

$$\Phi(\lambda, p(\lambda), q(\lambda), H_i(\lambda)) = 0 \tag{34}$$

By solving this equation, one finds $\lambda = \lambda(\sigma^{el}, H_i^{el})$ and, substituting it into Eqn (33), the final solution of the problem dependent only on elastic solution is obtained. Following can be considered as an example:

3.1 Von Mises Theory of Plasticity– Isotropic Hardening

This general scheme of the splitting method is applied to a specific type of the von Mises flow theory, with strain hardening. In this case, the plasticity condition (28) depends only on the second invariant of the stress deviator

$$\Phi = q - \sigma_Y - 2\mu_1 H = 0, \quad q = \sqrt{\frac{3}{2}\mathrm{s}:\mathrm{s}} \tag{35}$$

where q is the shear stress intensity, σ_Y is the yield limit, μ_1 is the shear hardening modulus, and H is the hardening parameter $\left(\dot{H} = \sqrt{\frac{2}{3}\dot{\varepsilon}^{pl}:\dot{\varepsilon}^{pl}}\right)$. The associated law and the stress relaxation Eqn (31) take the form

$$\begin{aligned} &\dot{\varepsilon}^{pl} = \dot{\lambda}\frac{d\Phi}{d\sigma} = \dot{\lambda}\frac{3}{2q}\mathrm{s} = \dot{\Lambda}\mathrm{s}, \\ &d\mathrm{s} = -2\mu\mathrm{s}d\Lambda, \ \dot{\Lambda} = \frac{3}{2q}\dot{\lambda} \end{aligned} \tag{36}$$

where μ is the shear modulus. Integrating Eqn (36) with the initial condition obtained after the elastic predictor $\Lambda = \Lambda_0$ and $s(\Lambda_0) = s^{el}$, one finds

$$\begin{aligned} &s = s^{el}\exp[-2\mu(\Lambda - \Lambda_0)], \\ &q = q^{el}\exp[-2\mu(\Lambda - \Lambda_0)] \end{aligned} \tag{37}$$

$$\dot{H} = \sqrt{\frac{2}{3}\dot{\varepsilon}^{pl}:\dot{\varepsilon}^{pl}} = \dot{\Lambda}\sqrt{\frac{2}{3}s:s} = \frac{2}{3}q\dot{\Lambda} \tag{38}$$

By integrating this equation with (37) taken into account, one obtains

$$\begin{aligned} H &= H^{el} + \frac{2}{3}q^{el}\int_{\Lambda_0}^{\Lambda}\exp[-2\mu(\Lambda' - \Lambda_0)]d\Lambda' \\ &= H^{el} - \frac{1}{3\mu}q^{el}\left\{\exp[-2\mu(\Lambda' - \Lambda_0)] - 1\right\} \end{aligned} \tag{39}$$

Denoting the correction coefficient by $x = \exp[-2\mu(\Lambda' - \Lambda_0)]$ and substituting the value H into the plasticity condition (35), we obtain the correction coefficient in the form

$$x = \frac{\sigma_Y + 2\mu_1 H^{el} + \frac{2\mu_1}{3\mu}q^{el}}{q^{el}\left(1 + \frac{2\mu_1}{3\mu}\right)} \tag{40}$$

Thus the stress at a point is equal to

$$\sigma = -pI + \mathrm{s} = -p^{el}I + \mathrm{s}^{el}x \tag{41}$$

where I is the identity tensor of rank 2, i.e., the solution of the elastoplastic problem can be obtained from the solution of the elastic problem simply by multiplication by the correction coefficient x. In the case of ideal plasticity, μ_1=0, one obtains

$$x = \frac{\sigma_Y}{q}, \quad s = \frac{s^{el}\sigma_Y}{q^{el}} \tag{42}$$

which means the well-known Wilkins correction rule of the stress tensor reduction to the yield surface in the stress space[12]. This rule holds only in the case of an ideal plastic medium and is incorrect for more complicated yield conditions, as it is clear from Eqn (40).

3.2 Numerical Results

As an example, we consider the shear problem for a bar in the three-dimensional case. A steel bar of dimensions $5 \times 2 \times 1$ mm is rigidly fixed at the endpoints. Its upper boundary moves with constant velocity applied in horizontal direction on the upper boundary. The shear displacement is equal to the bar thickness, i.e., 2 mm. The material dimensionless constants related to the yield limit are E = 500 and $\nu = 0.3$, and the modulus of plastic hardening is μ_1=0.1.

Figure 1(a) shows the bar strains, the deformation of the Lagrangian mesh, and the shear stress intensity

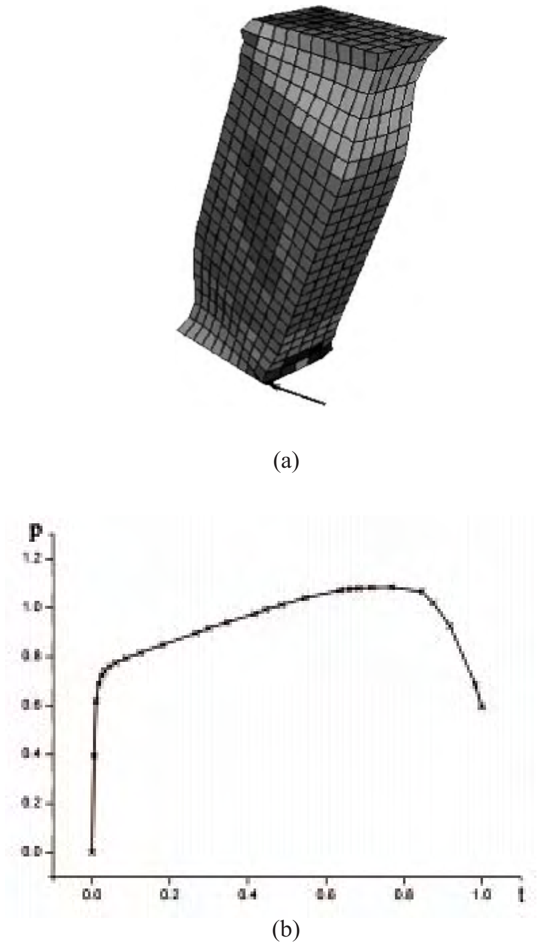


Figure 1. (a) Lagrangian mesh and (b) dependence of pressure *P* on loading parameter.

in the bar. Figure 1(b) presents the dependence of the pressure p on the loading parameter in the element marked by an arrow in Figure 1(a). The computations were performed with automatic step by two different methods. The solid line in Figure 1(b) corresponds to the computations with a small step by the method given in[13] and the lines marked by crosses correspond to the computations by the splitting method.

4. INTEGRATION OF CONSTITUTIVE RELATIONS FOR GTN AND MULTISCALE MODELS

The proposed splitting method has computational advantages over the standard iteration methods for solving the problem, because at each integration step the problem is reduced to solving an elastic problem at the predictor stage and to solving only one differential equation or a simple algebraic equation for the correction coefficient[14]. At the same time, if traditional methods are used, then we have to solve the system of $6+n$ constitutive equations at each point of the body, where n is the number of internal variables of the model.

4.1 Numerical Results. Boundary Problem

Consider the same shear problem for a bar in the three-dimensional case [Fig. 1(a)] solved for an elastoplastic material under the von Mises plasticity condition in 3.1. We present numerical results for the same boundary value problem in the case of a damaged medium satisfying the Gurson plasticity model (the GTN-model).

In the test example, the result obtained by the splitting method is in good agreement with the exact solution. The computations were performed by two different methods.

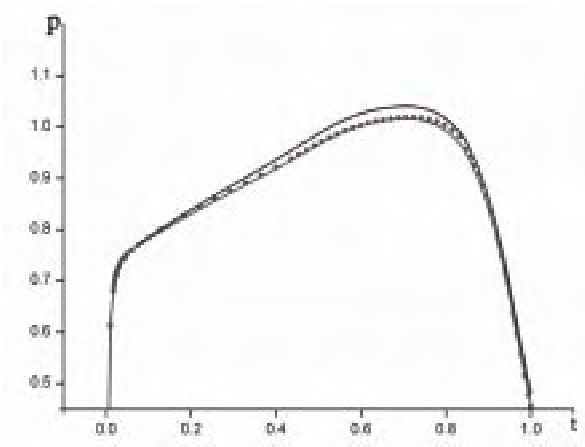
The curves 1 and 2 on Fig. 1(b) are obtained by the splitting method (with a fixed small step and an automatically chosen step by method given in[13] respectively). The error does not exceed 0.5 per cent.

The time per iteration in our method for integrating the constitutive relations is 2.4 times less than that in the method[13].

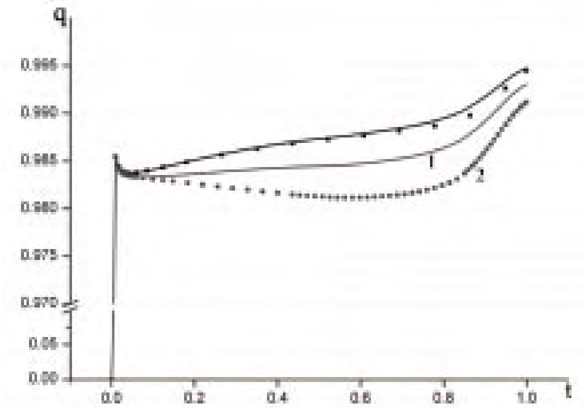
An advantage of this method is a faster rate of integration of the constitutive relations and simpler transformation of the Jacobian.

Comparing the results obtained in the above examples under the von Mises plasticity condition and the Gurson plasticity condition shows that the influence of the porosity on the stress-strain state is essential and must be taken into account.

In paper by Kukudzhanov[15], *et al.* and the present paper, this method is generalised to elastoplastic and elastoviscoplastic problems under a Gurson type plasticity condition with microvoids nucleation and growth taken into account. For the constitutive equations of the multiscale micromechanical model integration was done[16].



(a)



(b)

Figure 2. Dependence of the (a) pressure *P* and (b) shear stress intensity *q* on loading parameter in the element marked by the arrow in Fig. 1(a).

5. CONCLUSIONS

- An effective numerically-analytic method for integrating the elastoviscoplastic equations with internal variables is proposed. An advantage of this splitting method is that integrating a nonlinear system of n constitutive equations ($n = 6 + k$) for six components of the stress tensor and k internal variables in the case of coupled models of damaged plastic media is reduced to numerically solving a two nonlinear equations. The other equations can be integrated analytically, which makes the solution faster (more than twice for the GTN-model).
- The numerical-analytic solution of the system of constitutive equations allows easily analyzing the difference schemes for splitting the multiscale elastoviscoplastic problems and guarantees both the stability and the asymptotic convergence to the solution of the limit equilibrium equations as the small parameters δ_d , δ_p tend to zero[11,15,18].

ACKNOWLEDGEMENT

The research was supported by the Russian Foundation for Basic Research (grant no. 06-01-00523) and by program no. 14 of the Department of Energetics, Mechanical Engineering, Mechanics and Control Processes of the Russian Academy of Sciences.

REFERENCES

1. Gilliman, J.J. Dislocation dynamics and the response of material to impact. *Appl. Mech. Rev.,* 1968, **21**(8), 767-83.
2. Kukudzhanov, V.N. Micromechanical model of fracture of an inelastic material and its application to the investigation of strain localization. *Izv. AN. MTT* (*Mechanics of Solids*), 1999, **5**, 58-69.
3. Kukudzhanov, V.N., *et al.* On the problem of damage and localisation of strains. Chalmers Univ. Publ., Göteborg, 1995, Vol. 11, pp. 35.
4. Suzuki, T.; Eshinaga, H. & Takeuchi, S. Dynamics of dislocations and plasticity Mir, Moscow, 1989. (Russian translation).
5. Kukudzhanov, V.N. One-dimensional problems of the propagation of stress waves in rods. *In* Applied Mathematics Communications, No. 7, Computing Center of the USSR Academy of Science, Moscow, 1977. (in Russian).
6. Maiboroda, V.P.; Kravchuk, A.S. & Kholin, N.N. Velocity-driven deformation of materials. Mashinostroenie, Moscow, 1986. (in Russian).
7. Revnivtsev, V.I. (Ed.) Selective fracture of minerals. Nedra, Moscow, 1988. (in Russian).
8. Gurson, A.L. Continuum theory of ductile rupture by void nucleation and growth, Pt. I: Yield

criteria and flow rules for porous ductile materials. *Trans. ASME J. Engg. Mater. Technol.*, 1977, **99**(1), 2-15.

9. Chu, C.C. & Needleman, A. Void nucleation effects in biaxially stretched sheets. *Trans ASME J. Engg. Mater. Tech.*, 1980, **102**(3), 249-56.

10. Tvergaard, V. Influence of voids on shear band instabilities under plane strain condition. *J. Fract. Mech.*, 1981, **17**(4), 389-07.

11. Kukudzhanov, V.N. Decomposition method for elastoplastic equations. *Izv. RAN. MTT* (*Mechanics of Solids*), 2004, **1**, 98-108.

12. Wilkins, M. Calculations of elastic-plastic flow. *In* Methods of computational physics, edited by B. Adler, *et al.*, Vol. 3. Academic Press, New York, 1964.

13. Aravas, N. On the numerical integration of a class of pressure-dependent plasticity models. *Int. J. Numer. Methods Engg.*, 987, **24**(7), 1395-416.

14. Kukudzhanov, V.N. A new numerico-analytical method for the solution of elastoplastic equations based on the splitting of constitutive equations. *In* Computational plasticity, edited by D.R.T. Owen, *et al.*, Vol. 2, Barcelona CIMNE, 2005. pp. 711-14.

15. Kukudzhanov, V.N. A new splitting method for the solution of nonlinear equations of damaged elastoviscoplastic materials with complicated yield condition. *In* WCCM-VII, Los Angeles California, USA, 2006.

16. Kukudzhanov, V.N.; Levitin, A.L. & Synyuk, V.S. Numerical simulation of damaged elastoplastic materials (Preprint No. 807), *Int. Problem Mekhaniki RAN*, Moscow, 2006. (in Russian).

17. Synyuk, V. On the numerical integrtion of equations for coupled plasticity and damage models by a decomposition method. *In* EMMC-10 Conference Multi-phases and Multi-components Materials under Dynamic Loading, 11–14 June 2007, Kazimierz Dolny, Poland. pp. 291–03.

18. Kukudzhanov, Vladimir N. A new model and splitting method for the solution of equations of damaged elastoviscoplastic materials. *In* 9th International Symposium on Plasticity and Impact Mechanics IMPLAST 2007, 21–24, August 2007, Bochum (Germany). pp. 439–48.