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Finite Eulerian Elastoplasticity without Strain

A. Meyers, H. Xiao, and O. Bruhns

Ruhr-University Bochum, Germany

ABSTRACT

In the last few decades a number of phenomenological models have been developed for describing elastoplastic materials undergoing finite deformations. They are different in structure, e.g., their formulation is related to the reference or to the actual configuration , i.e., it is a Lagrangean or Eulerian formulation or it may contain elastic and/or plastic deformations. This study has shown that the most simple and straightforward obtained constitutive relation is free from any notion of elastic or plastic deformation. Moreover, it is related to the actual configuration, and thus omits to the greatest possible extent, quantities containing unwanted geometric deformation informations.

Keywords: Elastoplasticity, constitutive relations, Eulerian formulation, yield criterion, large deformations

NOMENCLATURE

B	Left Cauchy-Green tensor
B_i	i^{th} Eigenprojection of B
b_i	i^{th} Eigenvalue of B
C	Right Cauchy-Green tensor
D	Deformation rate tensor
D^e	Part of D related to recoverable work rate
D^p	Part of D related to dissipated work rate
F	Deformation gradient
f	Yield function
G	Evolution tensor of D^P
H	Fourth evolution tensor for D^P
h	Plastic modulus
J	Operator of objective time rate
I	Identity tensor
K	Second-order evolution tensor for k
k	Isotropic hardening variable
L	Velocity gradient tensor
p	Plastic potential
R	Rotation tensor
U	Right stretch tensor
V	Left stretch tensor
W	Vorticity tensor
w	Specific work
w^e	Specific elastic work
w^p	Specific plastic work
$\overline{w}$	Complementary potential
X	Position of material point in reference configuration
x	Position of material point in actual configuration

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α	Back stress; kinematic hardening tensor
ϕ	Fourth evolution tensor for D^p
Λ	Arbitrary Eulerian second-rank tensor
μ	Lamé constant
ν	Poisson ratio
Ψ	Fourth evolution tensor for α
σ	Cauchy or true stress tensor
τ	Kirchhoff stress tensor (τ = del $F\sigma$)
ξ	Plastic multiplier (1 on loading, 0 else)
$(.)^\circ$, $(.)^@$	Objective time rates
$(.)^{\circ\,(\log)}$	Logarithmic time rate

1. INTRODUCTION

Three-dimensional finite elastoplasticity for the description of isotropic materials has been primarily developed along the first-half of the last century; it gained a high level of maturity at the middle of that century[1-4]. Some of the major points seem to be essential are:

- Constitutive law is phenomenological, i.e., it is related to the macro-behaviour of the material.
- It is based on measurable physical quantities like deformation and forces and well established physical principles.
- It is observer-indifferent, i.e., objective.

A major challenge is to combine the elasticity, which is mostly relating strains and stresses, with the flow behaviour of plasticity. Initial attempts have been made by Hill[5] and Lehmann[6]; they based their theory on the additive decomposition of the deformation rate and used the hypoelastic law of grade zero. The stress rate was the Jaumann rate. The observation of simple shear stress oscillations[7-9] initiated a number of revised accesses to the domain of elastoplasticity, e.g., the use of other objective stress rates; the introduction of internal parameters like plastic strain or elastic and plastic deformation gradients.

Indeed, the number of objective stress rates is infinite. It may be noted that some of these eliminate the oscillation behaviour. However, the questions remain, which one of them is most appropriate and if unwanted behaviour is excluded for other typical computations?

This study shows, that there is an efficient and straightforward way to an Eulerian elastoplasticity that is simple in structure and free of any notion of elastic or plastic deformation.

2. KINEMATICS OF LARGE DEFORMATIONS

The X describes the position of a material point in the reference (Lagrange) configuration and x is the position in the actual (Euler) configuration. During a deformation of a continuum, neighbour points remain neighbour points. Hence, line element deformation and rotation are suitable quantities for identifying the body deformation. One has the following relations:

$$dx = F dX \tag{1}$$

$$F = \frac{\partial x}{\partial X} = RU = VR \;\; with\, R^{-1} = R^T, \quad \det R = 1,\; U = U^T, V = V^T \tag{2}$$

$$C = U^2 = F^T F,\; B = V^2 = FF^T \tag{3}$$

$$L = \dot{F} F^{-1} = D + W, \text{ with } D = \frac{1}{2}(L + L^T) = D^T,$$

$$W = \frac{1}{2}(L - L^T) = -W^T \tag{4}$$

For the material behaviour description Eulerian quantities are most suitable, since they are related to the deformed body. It may be shown that from the Eulerian quantities in Eqns (1)–(4), the stretch tensor V, the left Cauchy-Green tensor B, and the deformation rate D are objective.

While material time derivatives of scalars are objective, the objectivity of Eulerian tensor time derivatives is not as trivial. Known objective time rates of a second-order Eulerian tensor Λ can be represented in the form

$$\Lambda^\circ = \Lambda + \Lambda J + J^T \Lambda \qquad \text{where } J(F,\Lambda) \tag{5}$$

One may distinguish between corotational rates ($J^T = -J$), e.g.,

$J = W$ Zaremba/Jaumann rate[10,11]

$$J = \dot{R}/R^T \text{ Green/Naghdi rate}^{12} \qquad (6)$$

and non-corotational rates ($J^T \neq -J$), e.g.,

$J = \pm L$ Oldroyd rates[13]

$$J = -L + \frac{1}{2}\text{tr } LI \text{ Truesdell rate}^{14} \qquad (7)$$

The corotational logarithmic rate[15,16] will be of special relevance. For this rate, J may be written as function of the m distinct left Cauchy-Green tensor eigenvalues b_i and eigenprojections B_i as[17]

$$J = W + \sum_{i \neq k}^{m} \left(\frac{b_i + b_k}{b_i - b_k} - \frac{2}{\ln b_i - \ln b_k} \right) B_i D B_k \qquad (8)$$

3. STRAINLESS ELASTOPLASTICITY

3.1 Recoverable and Dissipated Energies

Let the specific work rate $\dot{w}$ be decomposed into a recoverable part $\dot{w}^e$ and a dissipated part $\dot{w}^p$, i.e.,

$$\dot{w} = \tau : D = \dot{w}^e = \dot{w}^p \qquad (9)$$

A possible access to Eqn (9) is the decomposition of the deformation rate D into the parts D^e and D^p

$$D = D^e + D^p \qquad (10)$$

3.2 Elastic and Plastic Behaviour

Since D^e describes the recoverable work rate part, it may be represented by an elastic rate form material law like Truesdell's hypoelastic relation[18]

$$D^e = H : \tau^{\circ} \qquad (11)$$

where $H(\tau)$ is a fourth-order tensor and $\tau = \det F\sigma$ is the Kirchhoff stress and the superscript circle stands for any objective rate. It will be seen that only a very restricted subset of Eqn (11) can serve the purposes here.

D^P is to be expressed by a flow-type law, e.g.,

$$D^P = \xi\Phi : \tau^{\circ}, \quad \alpha^{@} = \psi : D^P, \quad \dot{k} = K : D^P \qquad (12)$$

Here, $\Phi(\tau,\alpha,k)$ is a fourth-order tensor and ξ is the plastic multiplier taking the values 1 for loading and 0 for unloading. $\Psi(\tau,\alpha,k)$ and $K(\tau,\alpha,k)$ are fourth- and second-order tensors describing the evolution of the hardening parameters in function of the Kirchhoff stress τ, the kinematic hardening parameter α, and the isotropic hardening parameter k. $\alpha^{@}$ is an objective time rate of α.

It is assumed that there exists a yield function $f(\tau,\alpha,k)$ that limits the elastic or elastik-like range. Von Mises[19] proposed to introduce a plastic functional $p(\tau,\alpha,k)$ and to describe D^p by

$$D^p = \xi \frac{1}{h} \left(\frac{\partial f}{\partial \tau} : \tau^{\circ} \right) \frac{\partial p}{\partial \tau} \qquad (13)$$

Here, h is the plastic modulus. For $p = f$ relation Eqn (13) represents the associated flow rule.

3.3 Prager's Yielding Stationarity Criterion[20]

The simultaneous vanishing of τ°, $\alpha^{@}$ and $\dot{k}$ should render the yield surface $f(\tau, \alpha, k)$ stationary. From this criterion one may derive[21] that

- The stress rate τ° and the back stress rate $\alpha^{@}$ must be corotational, i.e., $J^T = -J$ must hold in Eqn (5).
- The stress and the back stress rates must be of the same corotational type.

3.4 Exact Integrability Condition

Elastic or elastic-like material behaviour occurs during the first loading in elastic range or during unloading. Then, $D^P = 0$ and the rate type elastic Eqn (11) remains. It was shown, however[22], that already for a very simple form of such relation, i.e., the hypoelastic relation of grade zero

$$2\mu D^e = \tau^{\circ} - \frac{\nu}{1+\nu} (\text{tr}\,\tau^{\circ}) I \qquad (14)$$

A number of known objective stress could not make the relation exactly integrable, i.e., make them fulfill Bernstein's integrability conditions[23,24]. Since the logarithmic rate was not yet disclosed, there study didn't include this rate.

In 1999 it was shown[25] that the rate equation

$$D^e = \frac{\partial^2 \overline{W}}{\partial \tau^2} : \tau^\circ \tag{15}$$

is exactly integrable to deliver a dissipationless elastic relation, if and only if

$$\tau^\circ = \tau^{\circ(\log)} \tag{16}$$

i.e., the objective stress rate in use is the logarithmic rate. By this the hypoelastic and hyperlastic formulations of elasticity have been brought together. It should be noted that Eqn (15) limits the available set of hypoelastic formulations Eqn (11).

3.5 Ilyushin's Postulate[26]

Let

$$D^e = \frac{\partial^2 \overline{W}}{\partial \tau^2} : \tau^{\circ(\log)} \tag{17}$$

and

$$D^p = \xi G(\tau^{\circ(\log)}, \tau, \alpha, k) \tag{18}$$

By a weakened formulation of Ilyushin's postulate, it has been shown[27] that

- the normality rule should hold, i.e.,

$$D^p = \xi \frac{1}{h} \left(\frac{\partial f}{\partial \tau} : \tau^{\circ(\log)}\right) \frac{\partial f}{\partial \tau} \tag{19}$$

and

- that the yield surface should be convex, i.e.,

$$(\tau - \tau_\circ) : \frac{\partial f}{\partial \tau} > 0 \tag{20}$$

4. STRAINLESS EULERIAN ELASTOPLASTICITY MODEL

In the previous sections it has been seen, that the stress and back stress rates should be of same corotational type (Prager's criterion), that the corotational rate should be logarithmic (exact integrability) and that the associated flow rule should hold (Ilyushin's postulate). This results in the following Eulerian model:

$$D = D^e + D^p = \frac{\partial^2 \overline{W}}{\partial \tau^2} : \tau^{\circ(\log)} + \xi \frac{1}{h} \left(\frac{\partial f}{\partial \tau} : \tau^{\circ(\log)}\right) \frac{\partial f}{\partial \tau} \tag{21}$$

The evolution laws of the kinematic and isotropic hardening parameters are

$$\alpha^{\circ(\log)} = \psi : D^p, \quad \dot{k} = K : D^p \tag{22}$$

From the plastic consistency condition ($\dot{f}$ = 0), one finds

$$h = \frac{-\partial f}{\partial \tau} : H : \frac{\partial f}{\partial \tau} - \frac{\partial f}{\partial k} \frac{\partial f}{\partial \tau} : K \tag{23}$$

Unified loading criteria for hardening and softening elastoplasticity may be written as

$$\begin{aligned} &\xi = 1 \ \text{if} \ f = 0 \ \text{and} \ \frac{\partial f}{\partial \tau} : \left(\frac{\partial^2 \overline{W}}{\partial \tau^2}\right)^{-1} : D > 0, \\ &\xi = 1 \ \text{if} \ f = 0 \ \text{and} \ \frac{\partial f}{\partial \tau} : \left(\frac{\partial^2 \overline{W}}{\partial \tau^2}\right)^{-1} : D \le 0, \\ &\xi = 0 \ \text{if} \ f < 0 \end{aligned} \tag{24}$$

The set of Eqns (21)–(24) represents a rate form elastoplasticity model. It is free from the notion of strain. Since no deformation decomposition in elastic and/or plastic parts has been performed, no evolution laws have to be formulated for the new quantities emerging from such a decomposition. It may be seen that theories with deformation decompositions, e.g., plastic strain or multiplicative deformation gradient decomposition, are restricted subsets of the present theory.

5. CONCLUSIONS

A simple and efficient Eulerian elastoplasticity model for isotropic materials is presented in a straightforward procedure. It is based on the decomposition of the work rate into recoverable and dissipated parts and is free from notions of elastic or plastic deformations. Only well established physical principles and strong mathematics lead to the final formulation. Hence, it is expected that it may be numerically most efficient and free of unwanted results.

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Contributors

Prof Heng Xaio studied Hyduralic and Electric Engineering in Wuhan (China) during 1983-86 and obtained his PhD from the Shanghai University. After completion of his post doctorate during 1990-1992 from the Xi' An Northwest University, he worked as Associate Professor at the Peking University from 1993-1995. He was Alexander-Von Homboldt Fellow at University of Bochum, Germany from 1995 to 1997, and after that, he stayed as scientific employee at the same university. He has published more than 70 research papers in different domains of continuum mechanics.

Prof Otto T. Bruhns studied Civil Engineering in Hannover (Germany) during 1962-67 and obtained his PhD in 1963. From 1975 to 1980, he worked as Professor at the University of Bochum, Germany; from 1980-1987, as full Professor in Kassel, Germany, and from 1987 up to now, as Full Professor in Bochum. He has published more than 170 scientific papers, 14 industrial reports, 10 monographs, and 10 books.