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Missile Longitudinal Dynamics Control Design using Pole Placement and LQR Methods – A Critical Analysis

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ABSTRACT

In high-maneuvering missile systems, with severe restrictions on actuator energy requirements, it is desirable to achieve the required performance with least actuation effort. Linear Quadratic Regulator (LQR) has been in literature for long and has proven it's mettle as an optimal controller in many benign aerospace applications and industrial applications where the response times of the plant, in most cases, are seen to be greater than 10 seconds. It can be observed in the literature that LQR control methodology has not been explored enough in the tactical missile applications where requirement of very fast airframe response times are desired, typically of the order of milliseconds. In the present research, the applicability of LQR method for one such agile missile control has been critically explored. In the present research work, longitudinal dynamic model of an agile missile flying at high angle of attack regime has been established and an optimal LQR control solution has been proposed to bring out the required performance demanding least control actuator energy. A novel scheme has been presented to further optimise the control effort, which is essential in this class of missile systems with space and energy constraints, by iteratively computing optimal magnitude state weighing matrix Q and control cost matrix R. Pole placement design techniques, though extensively used in aerospace industry because of ease of implementation and proven results, do not address optimality of the system performance. Hence, a comparative study has been carried out to verify the results of LQR against pole placement technique based controller. The efficacy of LQR based controller over pole placement design techniques is successfully established with minimum control energy requirement in this paper. Futuristic high maneuvering, agile missile control design with severe space and energy constraints stand to benefit incorporating the controller design scheme proposed in this paper.

Keywords: Optimal control; Longitudinal flight dynamics; Full-state feedback systems; Pole placement; Linear quadratic regulator

NOMENCLATURE

LQR	Linear Quadratic Regulator
u, v, w	Components of missile velocity three axes
p, q, r	Roll rate, Pitch rate and Yaw rate
θ	Pitch angle
PID	Proportional-Integral-Derivative
F_x, F_y, F_z	Components of total force along three axes
L, M, N	Rolling moment, Pitch moment, Yawing moment
m	Missile mass
Iy	Moment of Inertia about y-axis
AoA	Angle of Attack
ζ	Damping coefficient
MIMO	Multi-Input Multi-Output
SISO	Single Input Single Outupt
J	Cost function Jacobian
ARE	Algebraic Riccati Equation
DoF	Degree of Freedom
PP	Pole Placement

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1. INTRODUCTION

Neutralising the target with high accuracy and precision is highly essential in high maneuvering tactical agile missiles. Optimal control solution to modern missile design has become essential due to fact that there has been a paradigm shift in the missile design philosophies. Future requirements in state-of-the-art missile design calls for high performance in terms of g-maneuvers, high precision attacks, high angle of attack operation within stringent space, volume and power requirements. Though lot of literature has been found with applications in civil aircraft and commercial applications, very little practical applications are seen in tactical missile design domain.

Longitudinal dynamics (u, w, q, θ) are the dominant states while considering the g-maneuvers and the trajectory. In missile design with axi-symmetric configurations, longitudinal and lateral modes can be decoupled and independently studied. In this paper, the longitudinal dynamics of one such agile missile application has been investigated. Main objective in design of the tactical missile is to pull high g-maneuvers while keeping all missile states at trimmed condition, especially the longitudinal state parameters. Thus, longitudinal dynamics of one such missile is considered in the present study.

Human beings relentless persuasion for optimality of a solution is not new and this persuasion goes back till 300 years. In a historical perspective, it was Professor Johann Bernoulli, during 1695-1705, first published his solution of *'Brachysfochrone Problem'*[1]. Optimal solution in the field of aerospace industry is more relevant as we are heading into another decade, where we are stretching the boundaries of aerospace vehicles in extreme flight regimes, be it in the field of missiles, fighter aircrafts and civil airliners.

Fast forwarding 300 years later, the origin of optimal control theory can be traced to the work on the *'Pontryagin maximum principle'*[2] carried out by L.S. Pontryagin, in sixties which was primarily intended for military applications.

In missile applications, classical controllers such as PI, PD, PID[29-31] have been largely successful in autopilot design of SISO based systems due to their simplicity in implementation and tuning procedures.

Proportional action in PID speeds up the overall response of the system and also helps in reducing the steady-state error but suffers from disadvantages of offsets and maximum overshoots. Integral action in PID tends towards instability due to it's slow response in negating the error. Derivative controller in PID, though improves the transient characteristics, sometimes tends to produce saturation effects and amplifies in presence of noisy signal. Thus, PID controllers cannot guarantee the optimal performance of system with least control effort (a prime requirement of autopilot design in tactical missile system) and would be a poor choice when dealing with multi-state control system. Classical controllers do not address the MIMO plant and optimality requirement. Improvements to these stated drawbacks of the classical methods were addressed to a great extent in pole placement (PP) control design technique.

PP design philosophy[23,25], also known as full-state feedback (FSF), unlike classical methods, ensures that not only the dominant poles, but all poles lie at specified desired locations. Though, PP design technique, unlike classical methods, addresses the plant as a whole, it does not comply with optimality.

LQR control strategy addresses optimality requirement of MIMO plant and hence effort has been made to establish a LQR based controller to meet all the autopilot control demands requiring least control energy.

Comparison of LQR with classical controllers, such as PD controller has been discussed with application to inverted pendulum problem[3]. Similar study has been carried out comparing full-state feedback method, i.e. pole placement and LQR in controlling the inverted pendulum[4]. It was shown in[3,4] that LQR provides better results compared with FSF controller and was concluded that selection of cost function parameters, Q & R, can be adjusted using heuristic techniques for better results. However, in both[3,4], the results were based on classical problem, though it is a very interesting one to study for non-linear control system engineers.

Alandoli[5], *et al* carried out a comparative study with PID and LQR controllers for position tracking and vibration suppression of flexible link manipulator. The simulated results showed the capability of LQR controller for vibration suppression were better than PID controller. Similar study on comparison of pole placement with LQR control strategies has been carried out in[6] with application to single-link flexible manipulator. It was brought out that it was possible in practice to conserve material and energy resources using LQR for many high speed automation applications. The application of these techniques to a system that has one-degree of freedom was discussed in[6], whereas in missile and aerospace applications one has to deal with system of 6-DoF, which is complex in nature with its inherent non-linearities and cross-coupled aerodynamic and control coefficients. It can be inferred from both[5,6] that LQR provides a better solution. A comparative study among PID, PP and LQR controllers for heat exchanger application was carried out in[7]. The step response showed the adequacy of all the three control methods but LQR method performed better than PID and pole placement in terms of faster response and minimising system overshoots. Longitudinal control of missile has been carried out by various researchers using different modern control techniques[8-10] such as H∞[11], back stepping[12-14], sliding model control[15]. An application of LQR technique for UAV was presented in[16,17]. Flight dynamics of a damaged asymmetric aircraft using LQR was carried out to regain the aircraft stability was presented in[18]. Here, the flight dynamics were considered for the study and it was presented that LQR method would give better transient performances.

Based on the extensive literature survey carried out, it can be observed that LQR control design approach with minimum control energy requirement for missile applications with stringent design constraints has not been explored enough. In tactical missile applications , there are severe restrictions on size, weight and power requirements. Hence, there is a requirement to design a control system to achieve the required performance expending minimum control energy.

The main advantages of reducing the control energy in a missile system are: (i) desired flight performance and manoeuvres can be obtained with small tail control surfaces, (ii) saving of on-board power/energy for the demanded control deflection.

The main motivation of this research work is to develop an autopilot for such an ultra-compact agile missile application. Research work towards meeting these critical requirements will enable futuristic ultra-compact defence technologies.

In this paper, a successful attempt has been made to apply LQR control design strategies to a highly maneuverable missile longitudinal dynamics and the controller adequacy has been established. The novelty of the study has been brought out by establishing the missile longitudinal dynamics for this particular application and designing an optimal controller for the highly maneuvering missile system.

In order to compare the efficacy of LQR design method, the industry proven pole placement design methodology has also been implemented. The paper brings out through an exhaustive study, a critical qualitative and quantitative analysis of pole placement vs optimal LQR control design methodology.

As a generic case study, design of optimal control system with settling time not exceeding 500 ms and system overshoot less than 2% has been considered as design specification requirement for the present work.

2. MISSILE MATHEMATICAL MODEL

A representative missile 6-DoF model is shown in Fig. 1. Mathematically, the resultant missile velocity, $\vec{V}_m$, and angular velocity, $\vec{\omega}$ can be expressed as Eqn (1)[19,20].

$$\vec{V}_m = u\hat{i} + v\hat{j} + w\hat{k}\,;\quad \vec{\omega} = p\hat{i} + q\hat{j} + r\hat{k} \tag{1}$$

From Newton's second law, the equations of motion of a rigid-body are defined as Eqn (2)[20-22]:

$$\sum F = \frac{d}{dt}(mV_m)\bigg|_I\,;\ \sum M = \frac{dH}{dt}\bigg|_I \tag{2}$$

symbol $\big|_I$ indicates the time rate of change of the vector with respect to inertial space.

The above equations can be expressed in terms of three forces and three moments as in Eqns (3) and (4):

$$F_x = \frac{d(mu)}{dt}\,;F_y = \frac{d(mv)}{dt}\,;F_z = \frac{d(mw)}{dt} \tag{3}$$

and

$$L = \frac{dH_x}{dt}\,;M = \frac{dH_y}{dt}\,;\ N = \frac{dH_z}{dt} \tag{4}$$

The force equation expressed in body axes form as Eqn(5):

$$F = m\left[\frac{dV_m}{dt}\right]_{body} + m(\vec{\omega}\times\vec{V}_M) \tag{5}$$

After substituting $\vec{V}_M$ and $\vec{\omega}$, the translational equations of linear motion are obtained as shown in Eqn (6):

$$\begin{aligned} F_x &= m(\dot{u} + wq - vr)\\ F_y &= m(\dot{v} + ur - wp)\\ F_z &= m(\dot{w} + vp - uq) \end{aligned} \tag{6}$$

The moment of momentum can be expressed as in Eqn (7):

$$H = \sum rdm \times Vcm + \sum\left[r\times(\vec{\omega}\times r)\right]dm \tag{7}$$

where $\sum rdm = 0$.

Since, $\sum \Delta M = 1_H\left(\dfrac{dH}{dt}\right) + \vec{\omega}\times H$

where, $H = \int r\times(\vec{\omega}\times r)dm$

Substituting in the above equations we obtain the equations of rotational motion as given in Eqn (8):

$$\begin{aligned} L &= I_{xx}\dot{p}\\ M &= I_{yy}\dot{q} + (I_{xx} - I_{zz})pr\\ N &= I_{zz}\dot{r} + (I_{yy} - I_{xx})pq \end{aligned} \tag{8}$$

2.1 Linearisation of the Non-linear Plant

Eqn (6) and Eqn (8) form the three translational and three moment equations of the full 6-DoF missile dynamics of a rigid body. The equations of motion (EoM) are expressed in terms of

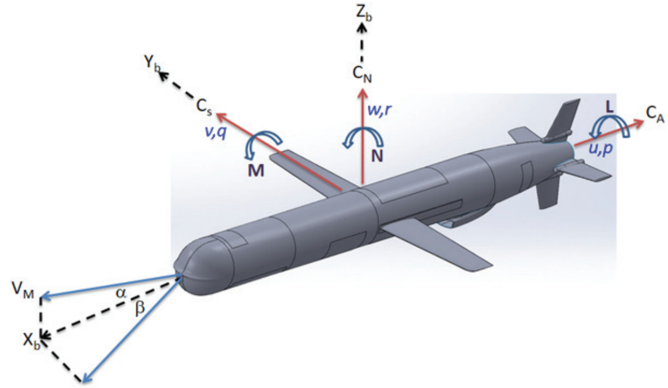

Figure 1. Missile 6-DoF model.

disturbing forces and moments due to i) aerodynamic effects, ii) gravitation effects, iii) deflections of aerodynamic controls, iv) power effects and v) effects of atmospheric disturbances. Therefore, the forces of Eqn (6) can be rewritten as shown in Eqn (9)

$$\begin{aligned} F_x &= X_{aero} + X_{gr} + X_{con} + X_{po} + X_{dis}\\ F_y &= Y_{aero} + Y_{gr} + Y_{con} + Y_{po} + Y_{dis}\\ F_z &= Z_{aero} + Z_{gr} + Z_{con} + Z_{po} + Z_{dis} \end{aligned} \tag{9}$$

Assuming the aerodynamic force and moment terms are dependent on disturbed motion variables and their derivatives only, the aerodynamic force can be expressed mathematically in terms of Taylor's series involving system states and their derivatives. Further, the expression can be simplified as follows:

$$\begin{aligned} X_{aero} &= X_{aero} + X_u u + X_v v + X_w w + X_p p + X_q q + X_r r + X_{\dot{w}}\dot{w}\\ L_{aero} &= L_{aero} + L_u u + L_v v + L_w w + L_p p + L_q q + L_r r + L_{\dot{w}}\dot{w} \end{aligned}$$

Since, the longitudinal EoM depend only on longitudinal parameters and hence decoupling them from lateral, EoM given in Eqn (6) result into following Eqn (10)[21,22].

$$\begin{aligned} m\dot{u} &= X_u u + X_w w + X_q q - mg\theta\cos\theta_e + X_\eta\eta\\ m\dot{w} &= Z_u u + Z_w w + Z_q q - mg\theta\sin\theta_e + Z_\eta\eta\\ I_y\dot{q} &= M_u u + M_w w + M_q q + M_\eta\eta\\ \dot{\theta} &= q \end{aligned} \tag{10}$$

where $X_u = \dfrac{\partial X}{\partial u}\bigg|_{u_{trim}}$ and so on.

3. MISSILE PLANT DYNAMICS

A yaw-to-turn missile configuration[23,24] was chosen for establishing the plant dynamics and the aerodynamic dimensional derivative data has been used in modelling the missile configuration. The data presented are from in-house semi-empirical prediction codes and wind tunnel aerodynamic data for this configuration. All aerodynamic coefficients are referenced to body cross-sectional area and diameter.

Here, the system dynamics are represented in state-space form as shown in Eqn (11):

$$\begin{aligned} \dot{x} &= Ax + Bu\\ y &= Cx + Du \end{aligned} \tag{11}$$

In the present study, the nonlinear plant is linearised about the trim conditions and matrices A and B are formed with constant aerodynamic coefficients. State-space approach is considered for designing the above dynamic model. From the longitudinal dynamics of Eqn. (10), the missile plant can be expressed in terms of state variables as shown in Eqn (12)[25-28].

$$A=\begin{bmatrix} -0.0089 & -0.1474 & 0 & -9.75 \\ -0.0216 & -0.3601 & 5.9470 & -0.151 \\ 0 & -0.00015 & -0.0224 & 0.0006 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B=\begin{bmatrix} 9.748 \\ 3.77 \\ -0.034 \\ 0.01 \end{bmatrix}$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

The variables of Eqn (12) are functions of u, w, q and θ. The missile dynamic model considered in Eqn (12) are computed at condition M = 0.7, AoA α = 15°, MSL and aerodynamic coefficients are as given in Eqn (13). The matrices of Eqn (13) correspond to A, B, C and D of Eqn (11).

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_u & x_w & x_q & x_\theta \\ z_u & z_w & z_q & z_\theta \\ m_u & m_w & m_q & m_\theta \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} x_\eta \\ z_\eta \\ m_\eta \\ 0 \end{bmatrix} \eta \tag{13}$$

3.1 Stability Analysis

Investigation of system properties, which indicate acceptable system performance is the first step in the process of control system design. In order to study the stability, the system is subjected to step response to find the initial transients to a pre-defined disturbance. The system is subjected to min-max disturbance ranges experienced during missile flight. Various disturbances, typically experienced in actual flights, have been simulated and the plant was found to be unstable. The response to one of the typical cases is shown in Fig. 2. The system open-loop (OL) response has been observed against an initial disturbance in all the states, namely, u, w, q and θ of 5 m/sec, 1m/sec, 1°/sec and 1° respectively. The system parameters as seen in Fig. 2 are not converging in finite time and found to be unstable.

3.2 Eigenvalues of Plant

It is observed from Fig. 2, that the forward velocity u increases with respect to time and is uncontrollable which leads to instability of the system. The system eigenvalues are computed to confirm the same and given in Table 1.

Table 1. Open-Loop Plant damping response characteristics

Pole	Damping	Frequency (rad/sec)	Time constant (sec)
-0.366	0.98	0.366	2.74
-0.622	0.96	0.062	1.61
0.182 + 0.03i	-0.49	0.037	-55.1
0.182 - 0.03i	-0.49	0.037	-55.1

For any stable system, as we know, all the roots of control system (eigenvalues) must lie in left half of s-plane (LHP)[29-31]. The eigenvalues / poles located away from the origin in LHP require more effort:

(a) high quality sensors for measurement to control the system, which in turn calls for and
(b) Large motors to drive the control surfaces leading to cost overruns.

As seen in Table 1, the system OL poles or eigenvalues of the plant are complex conjugate and one pair of poles [p = 0.182 $\pm$ 0.03i] are in right-hand side of the s-plane (RHP). The pole locations are also an indication of the eigenvalues of the plant. The damping values associated with the eigenvalues also suggest that they are negative (ζ = -0.49). Such systems will not be able to meet the system objectives, i.e., in case of missile system, it may overshoot or undershoot and will not be able to follow the planned trajectory and miss the target. Hence, it requires augmentation in design of the system damping and position of the eigenvalues to meet the system performance. Two other real poles are lying in the LHP and heavily damped. It is a prerequisite to examine the controllability of the plant before embarking on controller design, which is further discussed in the following section.

3.3 Controllability of a Plant

The concept of controllability was introduced by Kalman[32-34] applicable for MIMO systems. For any system, if it is possible to take it from any initial state $x(t_0)$ to any final state $x(t_f)$ in a finite time ($t=t_f-t_0$) by means of input vector u, it is said to be controllable system. Generally, an unstable close-loop pole can be cancelled with a zero to make the system stable. But, in some cases, this results into unstable close-loop (CL) system. Hence, the sufficient condition for complete state controllability matrix as defined in Eqn (14):

$$P=(B, AB, A^2B, ..., A^{n-1}B) \tag{14}$$

where P is the rank of the matrix. Since, the rank of the system, P is found to be equal to the number of states, 4 and hence the system is completely controllable.

The condition number i.e., the determinant of the controllability test matrix, *det (P)* defines the degree of controllability. Since, in this case, $det\ (P) \neq 0$ which is also an indication that the plant is controllable. However, the condition number of missile plant is high (~17200) suggests that the plant is weakly controllable.

Various techniques exist today in modern control system design to stabilise an unstable MIMO plant. Pole placement is one of such proven methods[35]. The following section presents the pole placement control design method in an attempt to stabilise the unstable missile plant as shown in Fig. 2.

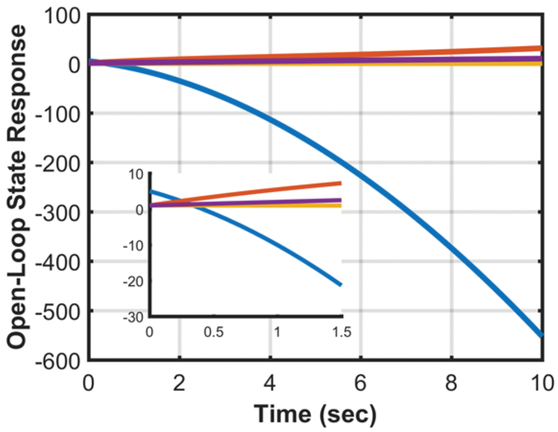

Figure 2. Open-Loop response of the system to Initial Disturbance.

4. POLE PLACEMENT CONTROL DESIGN TECHNIQUE

Pole placement method has been applied to stabilise the non-linear system dynamics through Ackermann's formula[36].

4.1 Pole placement design philosophy

Pole placement can be considered as a fancy root locus with an objective to find the controller gain matrix. Root locus concerns the gain change along the loci, whereas in the pole placement the designer has the flexibility in choosing dominant poles, i.e., close-loop eigenvalues of the plant. The block diagram of pole placement is shown in Fig. 3.

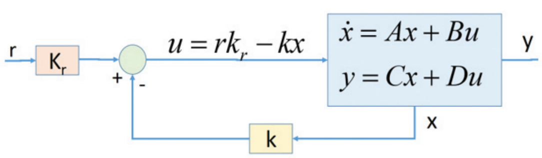

Figure 3. Pole placement block diagram.

In pole placement technique the state matrix is transformed as shown below:

$$\dot{x} = Ax + B(rk_r - kx)$$
$$\dot{x} = (A - Bk)x + Brk_r$$

(A-Bk) forms the CL plant model. Pole placement techniques implemented using Ackermann's formula and Butterworth method are presented in the following subsections with simulation results.

4.2 Simulation Results of Pole Placement

A perturbation study of close-loop poles over a range of values has been conducted and the initial values of the close-loop poles are chosen in order to avoid aggressive inputs and/or loop sensitivity. The CL poles are located at [s = -1$\pm$i and s = -5$\pm$5i]. The characteristic polynomials for the OL and CL poles are as given below:

a) Open-loop: $0.3914s^3 + 0.0086s^2 - 0.0002s + 0.0 = 0$

b) Close-loop: $39s^3 + 0.0768s^2 + 0.8819s + 5.0625 = 0$

The controller gain matrix '*K'* is computed from the upper triangular matrix[24]. Since, the gain matrix *K* does not provide the desired result, in the next iteration, the desired pole location is selected at $V = 0.5*[-0.5 \pm i, -1 \pm 2i]$. As seen in Fig. 4, the system response has largely improved from an unstable (Fig. 2) to stable state. However, oscillating states in u and w are observed which are not yet settled even after 10 seconds. High oscillations were observed in these two states. The system response has been analysed for 10 seconds. The missile under consideration is disturbed from its initial value with u = 5 m/sec, w = 1m/sec, q = 1°/sec and θ = 1° respectively. The close-loop plant is monitored with the initial disturbance with the pole-placement controller in place in the feedback are hardly disturbed in the dynamics even with the disturbances induced in the system. Therefore, it is evident from the above analysis, that the pole placement design technique is based on 'trial and error' method and this iterative method of placing the poles may be continued until a desired response is achieved which is a time consuming process.

The new controller gain elements are several times higher to stabilise the system and achieve a satisfactory settling time. The input $u = -Kx(t)$ is computed and plotted in Fig. 5.

The controller gain matrix computed based on Ackermann's formula will become inaccurate as the order of the plant increase, typically more than 10[36].

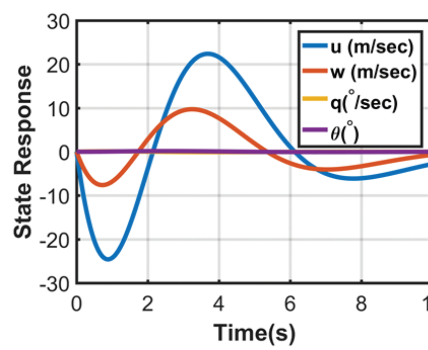

Figure 4. Close-Loop State response of the system to initial disturbance.

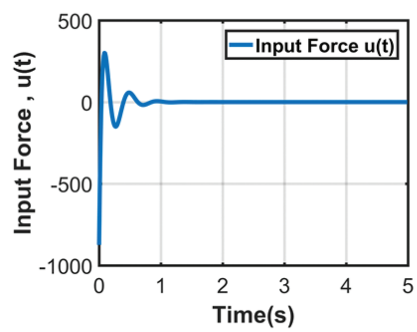

Figure 5. Input Force requirement for pole location at [s = 0.5*(-0.5$\pm$j, -1$\pm$2j)].

Any stable system requires that all CL poles are placed in LHP. If some of the CL poles are located near the imaginary axis and others farther, more control energy is required to achieve desired system performance. It can be observed from Fig. 5 that more effort is required with high initial requirement from the actuator to stabilise the airframe. In order to minimise the control effort Butterworth pattern[37] is adopted, in which poles are placed on a circle of radius 'R', centered at origin as given in Eqn (15).

$$\left(\frac{S}{R}\right)^2 = (-1)^{(n+1)} \tag{15}$$

As a second attempt, Butterworth Pole placement design method is applied. As a consequence, it is seen from Fig .6, the settling time of the system has improved significantly. However, the maximum overshoot is still higher for the system. Any control system, especially, in space-constrained missile, has stringent requirement on actuator power required to make the desired deflection commands to maneuver the missile. It is also desirous to make minimum effort while achieving the required performance parameters.

The control input u is monitored as a function of time and it is seen that high actuation energy is required to stabilise the system as shown in Fig. 7. Sometimes, this high actuation energy may not be available in the missile system to deliver to the control surfaces. Hence, the poles are further relocated at [-8 ± 3i, -4 ± 4i] and the results are indicated in Fig. 7. As seen, the overshoot of the system has reduced by half and also the settling time improves as all the states which is seen to settle within 2.0 seconds. Actuation force requirement being a prime criterion, a comparison of the input force for both controller gain matrices K1 and K2 are made and shown in Table 2. As seen, there is a significant improvement in the control force requirement which lessens the burden on the actuator system as shown in Fig. 7. This gain K2 is arrived through pole-placement design techniques which can lead to realizable controller for the given missile plant, however, does not meet the terminal requirements.

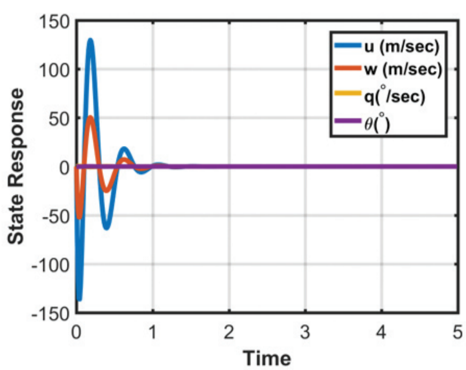

Figure 6. Close-loop state response of the system Butterworth with regulator gain K2.

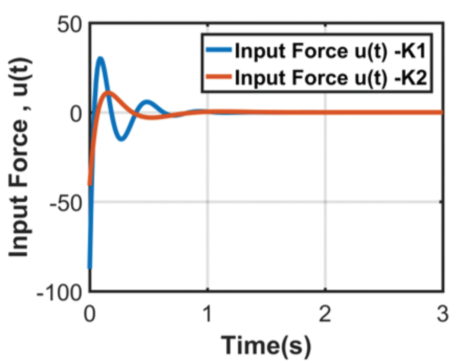

Figure 7. Input Force requirement for pole location at [s = -8 ± 3i, -4 ± 4i].

Table 2. Damping characteristics of the controller gain matrix K1 and K2

	Pole	Damping	Frequency (rad/sec)	Time constant (sec)
K1	-13.8 + 5.73i	0.92	15	0.07
	-13.8 - 5.73i	0.92	15	0.07
	-5.76 + 13.9i	0.38	15	0.17
	-5.76 - 13.9i	0.38	15	0.17
	Pole	**Damping**	**Frequency (rad/sec)**	**Time constant (sec)**
K2	-4 +4i	0.71	5.66	0.25
	-4 – 4i	0.71	5.66	0.25
	-8 + 3i	0.94	8.54	0.125
	-8 - 3i	0.94	8.54	0.125

The controller design through pole placement technique is presented in Table 2. However, in stark difference to pole-placement design methodology, the optimal control system directly addresses desired performance objectives, while minimising the control effort, which is highly desirable in a missile system.

Even though, pole placement has been a popular approach in aerospace applications, but is limiting in the intuitive understanding of the internal states of the system. Also, when optimality of the system requirements are concerned, there is a need to shift to more robust techniques in control system design. An optimal LQR design methodology has been carried out and presented in the following section.

5. LQR CONTROL DESIGN TECHNIQUE

The main idea in LQR control design philosophy is to minimise the quadratic Jacobian cost function matrix. This is discussed in detail in the following subsections.

5.1 LQR design philosophy

In optimal control, there are at least two elements, namely, i) the dynamics f consisting of the state variable x and control input u, as given in Eqn (16) and ii) the function J to be minimised given in Eqn (17)[25,26]. Optimal solution involves a minimisation of a function over a set of curves which itself is determined by some dynamical constraints.

$$\dot{x}(t) = f(x(t), u(t), t) \tag{16}$$

The cost function Jacobian, J can be written as shown in Eqn (17)

$$J = \frac{1}{2}\int_0^\infty [x^T Q_x x + u^T R_u u]dt \tag{17}$$

The objective of the optimal control problem is to find a control which causes the dynamical system to reach a target or follow a state variable and at the same time optimise the performance index J.

The technique for optimisation is discussed in following sections.

To optimise the performance index J, the state feedback regulator is designed. The gain matrix, K_r, is defined as per control law, Eqn (18)[25] for the plant described in Eqns (11) and (12)

$$u = -K_r x \tag{18}$$

subject to state equation constraints $\dot{x} = Ax + Bu$.

The critical design step in LQR design methodology is choosing the weights of Q_x and R_u. A priori knowledge of the open-loop system is required along with the specification of performance objectives to choose values for the cost function weight Q_x and R_u. The Q_x and R_u matrices can be selected using Bryson's rule[38]. The stability state space solution for K_r can be obtained as shown in Eqn (19).

$$K_r = R^{-1} B^T S \tag{19}$$

Solution for S can be obtained from Algebraic Riccati Equation (ARE):-

$$SA + A^T S - SBR^{-1} B^T S + Q_x = 0 \tag{20}$$

Controllability of plant *(A, B)* will suffice to obtain a unique solution to Eqn. (20). A regulator designed using this philosophy is called Linear Quadratic Regulator (LQR). The LQR diagram is shown in Fig. 8.

The open-loop response of the monitored states to step command is shown in Fig. 9 to the stated initial disturbance. The state parameter *u* diverges with respect to time. This is the dominant parameter which drives the system dynamics to instability. The other three parameters, *w*, *q* and θ are slightly unstable.

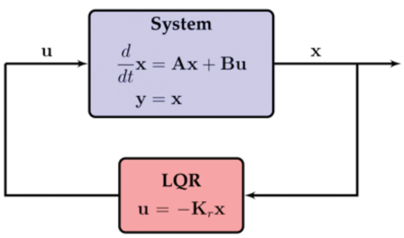

Figure 8. LQR controller schematic.

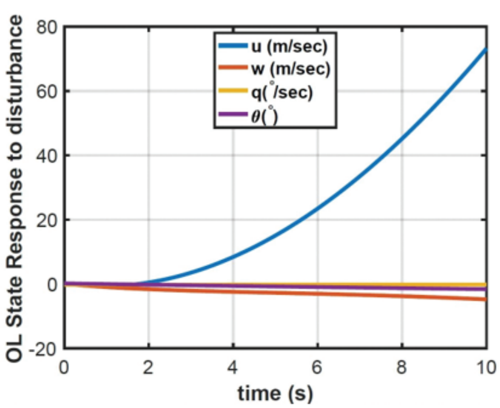

Figure 9. Open-loop state response to step command with initial disturbance.

5.2 Simulation Results of LQR

LQR control design methodology is introduced to this state of missile dynamics after formulation of the LQR control law shown in Eqn (18). The state response improves using LQR as shown in Fig. 10 but far away from desired performance index. Two of the states, *u* and *w*, have an oscillating tendency and do not converge. The other two states, *q* and θ are slightly unstable in the overall system dynamics after application of LQR design method in the feedback loop. The Control Law of LQR has been formulated with the following values of Q_x and R_u

$$Q_x = 0.1 * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}; R_u = [0.25]$$

The system is now triggered with a step command to monitor the transients and as seen from Fig. 11, and it is observed that the system is very lightly damped as the low frequency oscillations continue for a long time. However, a stark difference is noticed in the maximum overshoot when compared with results achieved through pole placement design techniques. The results obtained using LQR based design is able to bring down the system overshoots to a large extent. It is now to be seen on how to draw the fine line where acceptable levels of overshoot and settling time response is achieved with LQR based controller.

In the next iteration, the value of Q_x matrix is penalised with

$$Q_x = 0.1 * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 50 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 40 \end{bmatrix}; R_u = [0.25]$$

The step response is shown in Fig. 12. This further improves the system overshoot values when compared with

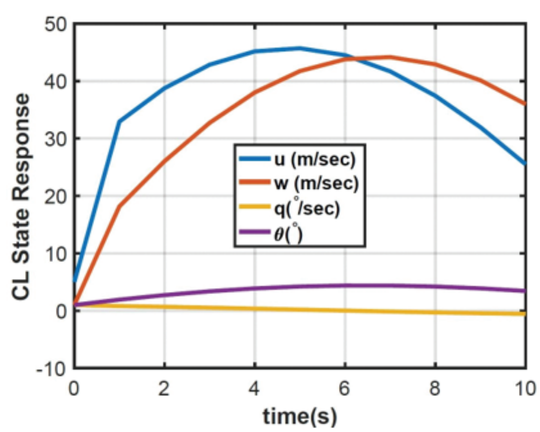

Figure 10. State response of system with LQR.

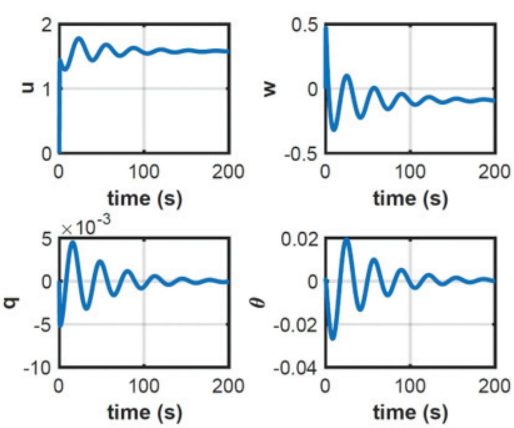

Figure 11. CL response using LQR with unit step command, Qdiag =0.1*[1 1 10 10].

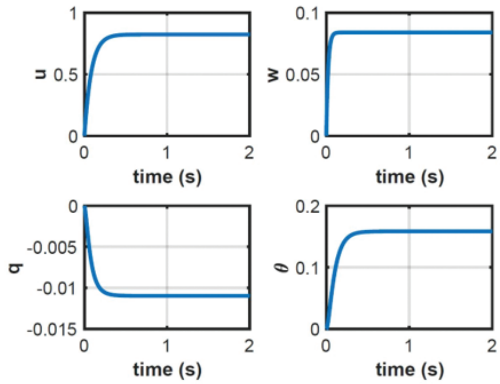

Figure 12. CL Response using LQR with unit step command, Qdiag = 0.1 * [1 50 10 40].

results obtained in Fig. 11. However, the system settling time is large and may not be acceptable when fast response is desired. A novel method is applied here to further improve the overall system performance. In order to control four variables (u, w, q, θ) independently to optimise the effort, R_u is constructed into a 4x4 vector form and Q_x is further penalised with 10 times weight.

The final Q_x and R_u matrices are arrived after running around 100 numerical simulations in an iterative manner. Each state weights has been changed while keeping the other state weights constant until the desired state response is achieved for each state within the specified settling time. A trade-off study between Q_x and R_u weights with the goal of minimising the cost function with design constraints has been carried out. Numerical simulation results highlighting various control effort requirement are shown in Fig. 14.

The constructed cost function takes the form of

$$\bar{Q}_x = 10 * Q_x \text{ and } \bar{R}_u = \begin{bmatrix} 0.55 & 0 & 0 & 0 \\ 0 & 0.45 & 0 & 0 \\ 0 & 0 & 0.35 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

As Q_x increases the state variables are affected and as R_u increases, the input activity gets lesser but state behaviour worsens. Hence Q_x is penalised with 10 times weight. As seen from Fig. 13, the system output states improve significantly and all states settle well within 360 msec.

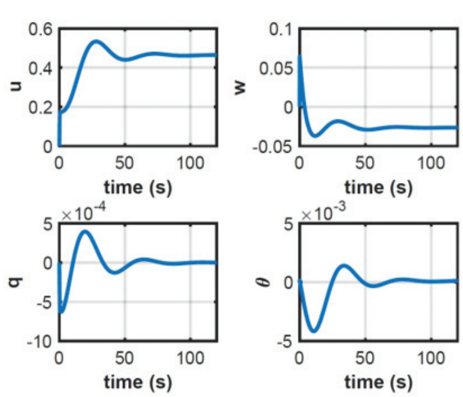

Figure 13. CL response using LQR with unit step command, Rdiag = [0.55 0.45 0.35 0.25].

The overshoot responses are attenuated with the chosen cost function and characterise a well-behaved damped system.

The controller gain matrix computed is $K_{LQR} = \left[2.40 \; 3.85 \; -586.36 \; -116.69\right]$.

Through LQR design methodology, this gain value has been evaluated and chosen for the highly maneuvering missile plant system.

It can be well stated that LQR control method when compared with pole placement method shows its superiority by bringing down the system overshoot and also alleviating the system transients much faster and providing excellent performance characteristics and applicable to systems where fast response with optimum control effort is desired.

6. RESULTS AND DISCUSSIONS

A realistic 6 DoF missile longitudinal dynamic plant has been established in state-space form, which, on analysing the open-loop response has been found to be unstable and weakly controllable.

The controller was designed using traditional pole-placement methodology. This design process was iterated to arrive at satisfactory performance from the close-loop system. Pole-placement technique, though, stabilises the system, does not give optimal performance in terms of system overshoots and settling time. In an attempt to improve the performance of the system, Ackermann's formula and Butterworth design were incorporated. Though the system performance significantly improved but it failed to meet the terminal requirements (Fig. 7).

The LQR control design methodology was implemented on the baseline missile configuration and it was found that it outperformed the pole placement performance indices.

Table 3 presents the quantitative analysis of the performance of both the controllers and as seen, implementing LQR controller has vastly improved the system performance characteristics.

In case of LQR, the settling time response is greatly improved, *viz.*, by about 9 times for *'u'* state and 20 times for '*w*'. It is also observed that the overshoot is completely nullified for all state variables.

The optimal and least control effort required to meet the desired trim condition of flight subjected to the initial disturbances is shown Fig. 14. Various costs of Q_x and R_u matrices have been chosen to arrive at the designed K_{LQR} controller gain matrix. The control deflection, an effective trim control deflection of δ_{pitch} = -9.76° (Fig. 14) is required to bring

Table 3. Comparison of performance of pole placement and LQR design techniques

State variable	Pole Placement			LQR		
	Rise Time	Settling Time	% OS	Rise Time	Settling Time	% OS
u	0.144	2.65	129.7	0.183	0.326	0.0
w	0.155	2.11	49.7	0.049	0.086	0.0
q	0.050	1.85	0.47	0.160	0.297	0.0
θ	0.038	1.75	0.14	0.197	0.356	0.0

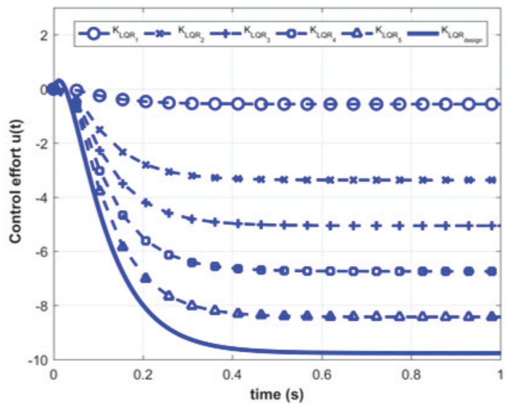

Figure 14. Control effort of various LQR design.

the airframe to a stable state, much within the actuator control limits. It can be observed from Fig. 7 and Fig. 14 that even after spending approximately 28% more actuator energy than LQR, PP technique is not meeting the terminal requirements.

The non-minimum phase behaviour is also noticed (t = 0 – 0.03 sec) in the response of the control effort as the fin deflection is seen to move in the opposite direction initially, a typical characteristic of tail-controlled missile.

7. CONCLUSION

The non-linear missile dynamic model derived forms the bedrock of the missile control design. The non-linear missile dynamics, presented in this paper, was defined in complete 6-DoF model and was further decoupled, trimmed and linearised about the operating point. The model formulated in the state-space form was used to synthesise pole placement and LQR control algorithms.

LQR based design incorporates cost function J in the feedback loop, where the plant dynamics and actuator are penalised with Qx and Ru respectively. In order to compare the efficacy of LQR design method, the industry proven pole placement method has been used for comparison studies.

Based on the numerical simulation results, it has been established that LQR based design performs better than pole-placement method in terms of state overshoot and settling time. The results indicate that the controller gain parameters for the plant are well established.

The novel approach presented in this paper can be used for design of autopilot for agile missiles with low energy requirements.

As part of future work, further investigation can be taken up on robust control design of missile system using LQR in presence of system noise & flight disturbances.

REFERENCES

1. Sussmann, H. J. & Willems, J. C. 300 years of optimal control: from the brachystochrone to the maximum principle. *IEEE Control Syst. Magazine,* 1997, **17**(3), 32-44. doi: 10.1109/37.588098
2. Bennett, S. A brief history of automatic control. *IEEE Control Systems Magazine*, 1996, **16**(3), 17-25. doi: 10.1109/37.506394
3. Singh, N. & Yadav, S. K. Comparison of LQR and PD controller for stabilizing Double Inverted Pendulum System. *Int. J. Engineering Res. Devel.*, 2012, **1**(12), 69-74.
4. Razmjooy, N.; Madadi, A.; Alikhani, H. R. & Mohseni, M. Comparison of LQR and Pole Placement Design Controllers for Controlling the Inverted Pendulum. *J. World's Electrical Eng. Technol.*, 2014, **3**(2), 83-88.
5. Alandoli, E. A.; Rashid, M.Z.A. & Marizan Sulaiman, A comparison of PID and LQR controllers for Position Tracking and Vibration Suppression of Flexible Link Manipulator, *J. Theoretical and Applied Information Technol.*, 2017, **95**(13), 2949-2955.
6. Saini, S. C.; Sharma, Y.; Bhandari, M. & Satija, U. Comparison of Pole Placement and LQR Applied to Single Link Flexible Manipulator, *In* Proceedings of International Conference on Communication Systems and Network Technologies, 2012, Rajkot, 843-847, doi: 10.1109/CSNT.2012.183
7. Mohamed Essahafi & Mustapha Ait Lafkih. Evaluation Performance of PID, LQR, Pole Placement Controllers for Heat Exchanger, *Int.J. Electr., Comput., Energetic, Electron. Commun. Eng.*, 2014, **8**(6), 939-943.
8. Lee, J; Cho, N & Kim, Y. Analysis of Missile Longitudinal Autopilot Based on the State-Dependent Riccati Equation Method. *J. Guidance, Control, Dynamics*. 2019, **42**(10), 2183-2196. doi: 10.2514/1.G003679
9. Zhang. Bo, Qixing. Lv & Yuqing Lei. The Application of PID Controller in Missile Longitudinal Loop System and Its Simulation, *In* Proc. of the 5th International Conference on Advanced Design and Manufacturing Engineering, Atlantis Press, Oct. 2015, 2113-2118, doi: 10.2991/icadme-15.2015.395
10. Xin, M. & Balakrishnan, SN. Missile longitudinal autopilot design using a new suboptimal nonlinear control method. *IEE Proceedings-Control Theory and Applications*, 2003, **150**(6) 577-84.
11. Mahmood, A.; Kim, Y. & Park, J. Robust H∞ autopilot design for agile missile with time-varying parameters. *IEEE Trans. Aerospace Electron. Syst.*, 2014, **50**(4), 3082-89. doi: 10.1109/TAES.2014.130750
12. Lee, CH. Kim, TH. & Tahk, MJ. Agile missile autopilot design using nonlinear backstepping control with time-delay adaptation. *Trans. Japan Society Aeronautical Space Sci.*, 2014, **57**(1), 9-20. doi: 10.2322/tjsass.57.9
13. Fan, J. F. & Su. Z., Missile longitudinal autopilot design using backstepping approach. *In* Proceedings of IEEE Aerospace Conference, Big Sky, MT, 2010, pp. 1-8. doi: 10.1109/AERO.2010.5446743
14. Li, MH.; Yin, J.; Sun, XY.; Xu, JX. & Zhang, MM. Design of Missile Longitudinal Control System Based on Backstepping Control. *Journal of Applied Mechanics and Materials,* 2014, 496–500, 1401–1406. doi:10.4028/www.scientific. net/amm.496-500.1401
15. Moon, G. & Lee, Y. A. Study on Missile Autopilot Design using Nonlinear Adaptive Sliding Mode Control Scheme. *In* Proceedings of 7th European Conference for Aeronautics And Space Sciences (EUCASS) July 3-6, 2017 - Milan

(Italy). doi: 10.13009/EUCASS2017-94
16. Nair, M. P. & Harikumar, R. Longitudinal dynamics control of UAV, *In* Proceedings of International Conference on Control Communication & Computing India (ICCC), Trivandrum, 2015, pp. 30-35, doi: 10.1109/ICCC.2015.7432865.
17. Putro, IE. & Duhri, RA. Longitudinal stability augmentation control for turbojet UAV based on linear quadratic regulator (LQR) approach. *In* Proceedings of AIP Conference, 2020, **2226**(1). doi:10.1063/5.0002786
18. Ogunwa, T. T. & Abdullah, E. J. Flight dynamics and control modelling of damaged asymmetric aircraft, *In* IOP Conference series: Material Science and Engineering 152, 2016, 012022. doi:10.1088/1757-899X/152/1/012022
19. Roskam, Jan. Airplane Flight Dynamics and Automatic Flight Controls. 1. Ottawa, Kansas: Roskam Aviation and Engineering Corporation. p. 113. Library of Congress Catalog Card Number: 78-31382
20. Babister, A. W. Aircraft Dynamic Stability and Response. Elsevier 1980, ISBN 0-08-024768-7
21. Etkin, B. Dynamics of Atmospheric Flight. John Wiley and Sons, Inc. New York, 1972.
22. Cook, M. V. Flight Dynamics Principles, 2nd Ed. Elsevier Aerospace Series, ISBN: 978-0-7506-6927-6
23. Friedland, B: Control System Design. McGraw-Hill Book Company 1987. ISBN 0-07-100420-3
24. George, M Siouris. Missile Guidance and control systems, Springer-Verlag, 2004.
25. Tewari, Ashish. Automatic Control of Atmospheric and Space Flight Vehicles, Birkhauser Publications, Control Engineering Series, ISBN-10: 0817648631, August 2011.
26. Tewari, Ashish. Modern Control Design, John Wiley & Sons, ISBN 0 471 496790, February 2002.
27. Jack, N. Nielsen. Missile Aerodynamics, McGraw Hill Series in Missile and Space Technology, 1960
28. ESDU 1987: Introduction to Aerodynamic Derivatives, Equations of Motion and Stability. Engineering Sciences Data Unit, Data Item No. 86021. Aerodynamics Series, Vol. 9a, Stability of Aircraft. Engineering Sciences Data, ESDU International Ltd.
29. Nagrath, I. J. & Gopal, M. Control Systems Engineering, 4th Ed. New Age International Publishers, 2006.
30. Farid Golnaraghi & Benjamin, C. Kuo. Automatic Control Systems, 9th Ed. John Wiley & sons inc., 2010.
31. Katsuhiko Ogata. Modern Control Engineering, Pearson Education Pvt Ltd., 2002.
32. Kálmán, R. Contributions to the Theory of Optimal Control. 1960.
33. Kalman, R. E. A New Approach to Linear Filtering and Prediction Problems. ASME. *J. Basic Eng*. 1960, **82**(1), 35-45. doi:10.1115/1.3662552
34. Kalman, R. On the general theory of control systems, *IRE Transactions on Automatic Control*, 1959, **4**(3), 110-110. doi: 10.1109/TAC.1959.1104873
35. Grewal, M. S. & Andrews, A. P. Applications of Kalman Filtering in Aerospace 1960 to the Present [Historical Perspectives], *IEEE Control Systems Magazine*, 2010, **30**(3), 69-78. doi: 10.1109/MCS.2010.936465
36. Ackermann, J., Der Entwurf linearer Regelungssysteme im Zustandraum, Regelungstechnik und Prozess-Datenverarbeitung, 1972, **7**, 297- 300.
37. Butterworth S., On the Theory of Filter Amplifiers*, Experimental Wireless & the Wireless Engineer*, Oct 1930, 536-541.
38. Bryson, A. E. Jr. Some Connections between Modern and Classical Control Concepts. ASME. *J. Dyn. Sys. Meas., Control*., 1979, **101**(2), 91–98. doi:10.1115/1.3426420

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