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Accuracy Analysis of Attitude Computation Based on Optimal Coning Algorithm

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ABSTRACT

To accurately evaluate the applicability of optimal coning algorithms, the direct influence of their periodic components on attitude accuracy is investigated. The true value of the change of the rotation vector is derived from the classical coning motion for analytic comparison. The analytic results show that the influence of periodic components is mostly dominant in two types of optimal coning algorithms. Considering that the errors of periodic components cannot be simply neglected, these algorithms are categorized with simplified forms. A variety of simulations are done under the classical coning motion. The numerical results are in good agreement with the analytic deductions. Considering their attitude accuracy, optimal coning algorithms of the 4-subinterval and 5-subinterval algorithms optimized with angular increments are not recommended for use for real application.

Keywords: Accuracy analysis, attitude computation, periodic components, optimal coning algorithms

1. INTRODUCTION

Attitude computation is a critical issue in the strapdown inertial navigation system (SINS). In the principal software function of the SINS, attitude computation is necessary for the transformation of acceleration which is twice integrated into velocity and position. Furthermore, noncommutativity of finite rotations is an inevitable phenomenon in the process of attitude computation. Therefore, how to design a special and efficient algorithm has been attracted over the past decades.

Many works appeared in the literatures describing the attitude computation based on the rotation vector proposed by Bortz[1]. In 1983, Miller[2] firstly developed an approach which updated the attitude quaternion with the change of the rotation vector during the updating period. The change of the rotation vector consisting of the outputs of incremental gyros was optimised under the classical coning motion[2] (called the optimal coning algorithm). Afterward, Ignagni[3-5] provided a convenient method to simplify Miller's derivation and made an assessment of the error inherent in the simplified form of the Bortz equation[1]. Several improved algorithms were also presented, additionally using the current and previous accumulated gyro outputs[6] or a higher-order term when the angular rates were known analytically[7]. To obtain the incremental signals involved in Miller's algorithm from the rate outputs of some modern-day gyros, hardware-based integrators are imperative. Addressing this issue, Huang and Deng[8] presented an improved form including angular increments and angular rates. But in these cases, the use of hardware-based integrators has contributed to increase the cost and complexity of system. To overcome this problem, Zeng[9], *et al*. stated a coning algorithm optimised with angular rates, which included a polynomial fitting procedure and a solution for coning compensation coefficients. Generalized method for this algorithm was detailedly introduced by Ben[10], *et al.*

Although the mentioned kinds of algorithm worked well to minimize the residual error of one nonperiodic component, the accuracy of attitude computation in the case of multi-subinterval was rarely discussed. In this study, we analyse the influence of periodic components in the optimal coning algorithms, mainly aiming to evaluate the accuracy of attitude computation based on each algorithm.

2. CLASSICAL CONING MOTION

The classical coning motion is a typical environment for the optimisation and effectiveness test of the coning algorithm, which can be characterized by the unit vector

$$\frac{\boldsymbol{a}(t)}{a} = \begin{bmatrix} 0 \\ \cos(\gamma t) \\ \sin(\gamma t) \end{bmatrix} \tag{1}$$

where a is the coning half-angle equaling the module of the vector $\boldsymbol{a}$, i.e., $a = (\boldsymbol{a}.\boldsymbol{a})^{1/2}$, $\gamma = 2\pi f$, and f is the coning frequency[1]. Based on the classical coning motion, the attitude quaternion $\boldsymbol{q}(t)$ for coordinate transformation can be formulated as

$$\boldsymbol{q}(t) = \begin{bmatrix} \cos\left(\frac{a}{2}\right) \\ \sin\left(\frac{a}{2}\right)\boldsymbol{a}(t) \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{a}{2}\right) \\ 0 \\ \sin\left(\frac{a}{2}\right)\cos(\gamma t) \\ \sin\left(\frac{a}{2}\right)\sin(\gamma t) \end{bmatrix} \tag{2}$$

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and the angular rate $\omega(t)$ representing the relative angular rate between two coordinate frames can be derived as[2]

$$\omega(t)=2\boldsymbol{q}^{*}(t)\circ\dot{\boldsymbol{q}}(t)=\begin{bmatrix}-2\gamma\sin^{2}\left(\frac{a}{2}\right)\\ -\gamma\sin a\sin(\gamma t)\\ \gamma\sin a\cos(\gamma t)\end{bmatrix} \quad (3)$$

3. ATTITUDE COMPUTATION

If only angular motion is considered, attitude quaternion can be simply updated as follows

$$\boldsymbol{q}(t+\Delta T)=\boldsymbol{q}(t)\circ\boldsymbol{q}(\Delta T) \quad (4)$$

where $\circ$ denotes quaternion multiplication, $\boldsymbol{q}(t+\Delta T)$ and $\boldsymbol{q}(t)$ are the attitude quaternions at time $t+\Delta T$ and time t, respectively, ΔT is the updating period of attitude quaternion, and $\boldsymbol{q}(\Delta T)$ is the quaternion representing the change of the attitude quaternion during time interval $(t,\ t+\Delta T)$, which can be obtained by[2]

$$q(\Delta T)=\begin{bmatrix}\cos\left(\frac{\Delta\sigma}{2}\right)\\ \frac{\Delta\boldsymbol{\sigma}}{\Delta\sigma}\sin\left(\frac{\Delta\sigma}{2}\right)\end{bmatrix} \quad (5)$$

where $\Delta\boldsymbol{\sigma}$ is the change of the rotation vector during time interval $(t,\ t+\Delta T)$ with module $\Delta\sigma=(\Delta\boldsymbol{\sigma}.\Delta\boldsymbol{\sigma})^{1/2}$. When $\Delta\sigma$ is near zero, power series expansions should be applied to the trigonometric function coefficients of Eqn (5) to avoid singularity. In this study, a fourth-order truncation is utilized

$$\cos\left(\frac{\Delta\sigma}{2}\right)=1-\frac{(\Delta\sigma)^{2}}{8}+\frac{(\Delta\sigma)^{4}}{384}$$
$$\frac{1}{\Delta\sigma}\sin\left(\frac{\Delta\sigma}{2}\right)=\frac{1}{2}-\frac{(\Delta\sigma)^{2}}{48} \quad (6)$$

The theoretical value of $\Delta\boldsymbol{\sigma}$ can be derived as Eqn (7) by integrating the Bortz equation[1] from time t to time $t+\Delta T$ and making some simplifications[3]

$$\Delta\bar{\sigma}=\sigma+\frac{1}{2}\int_{t}^{t+\Delta T}\sigma(\tau)\times\omega d\tau,\quad \omega(\tau)=\int_{t}^{\tau}\omega(\upsilon)d\upsilon \quad (7)$$

where $\boldsymbol{\omega}$ is similar to the angular rate described in Eqn (3), $\boldsymbol{\alpha}(\tau)$ is the instantaneous integration of $\boldsymbol{\omega}$ from time t, and $\boldsymbol{\alpha}=\boldsymbol{\alpha}(t+\Delta T)$. Considering the classical coning environment, Eqn (7) can be rewritten as

$$\Delta\bar{\sigma}=\begin{bmatrix}\Delta\bar{\sigma}_x\\ \Delta\bar{\sigma}_y\\ \Delta\bar{\sigma}_z\end{bmatrix}=\begin{bmatrix}-2\gamma\Delta T\sin^{2}\left(\frac{a}{2}\right)+\frac{1}{2}\sin^{2}a\left(\gamma\Delta T-\sin(\gamma\Delta T)\right)\\ -2\sin a\sin\left(\gamma\left(t+\frac{\Delta T}{2}\right)\right)\left(\sin\left(\frac{\gamma\Delta T}{2}\right)+\sin^{2}\left(\frac{a}{2}\right)\left(-\gamma\Delta T\cos\left(\frac{\gamma\Delta T}{2}\right)+2\sin\left(\frac{\gamma\Delta T}{2}\right)\right)\right)\\ 2\sin a\cos\left(\gamma\left(t+\frac{\Delta T}{2}\right)\right)\left(\sin\left(\frac{\gamma\Delta T}{2}\right)+\sin^{2}\left(\frac{a}{2}\right)\left(-\gamma\Delta T\cos\left(\frac{\gamma\Delta T}{2}\right)+2\sin\left(\frac{\gamma\Delta T}{2}\right)\right)\right)\end{bmatrix} \quad (8)$$

On the other hand, the quaternion $\boldsymbol{q}(\Delta T)$ can be calculated from Eqns (4) and (2)

$$q(\Delta T)=q^{*}(t)\circ q(t+\Delta T)=\begin{bmatrix}1-2\sin^{2}\left(\frac{a}{2}\right)\sin^{2}\left(\frac{\gamma\Delta T}{2}\right)\\ -\sin^{2}\left(\frac{a}{2}\right)\sin(\gamma\Delta T)\\ -\sin a\sin\left(\gamma\left(t+\frac{\Delta T}{2}\right)\right)\sin\left(\frac{\gamma\Delta T}{2}\right)\\ \sin a\cos\left(\gamma\left(t+\frac{\Delta T}{2}\right)\right)\sin\left(\frac{\gamma\Delta T}{2}\right)\end{bmatrix} \quad (9)$$

where * denotes conjugate operator[2].

For brevity, our discussions are confined to the single-speed-N-subinterval coning algorithm. Thus dividing time interval $(t,\ t+\Delta T)$ into N subintervals of equal width $\Delta t=\Delta T/N$, angular increments $\boldsymbol{\alpha}_i$ (i = 1, 2, …, N) and angular rates ω_i (i = 0, 1, …, N) can be acquired from Eqn (3) to optimise the following coning algorithms. Here Δt is just the sampling time.

3.1 Coning Algorithms Optimised with Angular Increments

Under a simplified coning motion, Ignagni[3] validated that the value of the cross product of two angular increments depended only on their spacing. A similar property can be derived under the classical coning motion, just referring to the nonperiodic component of $\alpha_i\times\alpha_j$. Using this property, the change of the rotation vector during time interval $(t,\ t+\Delta T)$ is approximated

$$\Delta\tilde{\sigma}=\sum_{i=1}^{N}\alpha_i+\sum_{i=1}^{N-1}K_i\left(\alpha_i\times\alpha_N\right) \quad (10)$$

where K_i (i = 1, 2, …, $N-1$) are constant coefficients. If $\gamma\Delta T<1$, the coefficients K_i can be optimised by using power series expansions and minimizing the error of the nonperiodic component between Eqns (8) and (10).

The optimal coning algorithms in the form of Eqn (10) are listed in Table 1, where N = 1, 2, 3, 4, 5. In this study, the absolute value of the residual error along nonperiodic component is defined as the error drift of algorithm (EDOA).

3.2 Coning Algorithms Optimised with Angular Rates

Considering the given rate signals, the definite integral of a fitted polynomial is used to determine the accumulated angular increment α_s during time interval $(t,\ t+\Delta T)$. Furthermore, noticing that $\omega_i\times\omega_j$ under the classical coning motion has a property similar to $\alpha_i\times\alpha_j$, the change of the rotation vector is approximately assumed as follows

$$\Delta\tilde{\sigma}=\alpha_s+(\Delta T)^{2}\sum_{i=0}^{N-1}M_i\left(\omega_i\times\omega_N\right) \quad (11)$$

where M_i (i = 0, 1, …, $N-1$) are constant coefficients. Finally, taking the same as what has been done in the derivation of K_i, the coefficients M_i can be optimally solved[9,10].

The coning algorithms in the optimal expression of Eqn (11) and the corresponding EDOAs are shown in Table 2, where N = 1, 2, 3, 4.

Table 1. Coning algorithms optimised with angular increments[4,11,12]

N	$\Delta\tilde{\sigma}$	EDOA (rad)
1	α_1	$\frac{(\gamma\Delta T)^3 \sin^2 a}{12}$
2	$\alpha_1+\alpha_2+\frac{2}{3}\alpha_1\times\alpha_2$	$\frac{(\gamma\Delta T)^5 \sin^2 a}{960}$
3	$\alpha_1+\alpha_2+\alpha_3+\frac{9}{20}\alpha_1\times\alpha_3+\frac{27}{20}\alpha_2\times\alpha_3$	$\frac{(\gamma\Delta T)^7 \sin^2 a}{204120}$
4	$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\frac{18}{35}\alpha_1\times\alpha_4+\frac{92}{105}\alpha_2\times\alpha_4+\frac{214}{105}\alpha_3\times\alpha_4$	$\frac{(\gamma\Delta T)^9 \sin^2 a}{82575360}$
5	$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\frac{125}{252}\alpha_1\times\alpha_5+\frac{25}{24}\alpha_2\times\alpha_5+\frac{325}{252}\alpha_3\times\alpha_5+\frac{1375}{504}\alpha_4\times\alpha_5$	$\frac{(\gamma\Delta T)^{11} \sin^2 a}{54140625000}$

Table 2. Coning algorithms optimised with angular rates[9,10]

N	$\Delta\tilde{\sigma}$	EDOA (rad)
1	$\frac{\Delta T}{2}(\omega_0+\omega_1)+\frac{(\Delta T)^2}{12}(\omega_0\times\omega_1)$	$\frac{7(\gamma\Delta T)^5 \sin^2 a}{720}$
2	$\frac{\Delta T}{6}(\omega_0+4\omega_1+\omega_2)+\frac{(\Delta T)^2}{180}(\omega_0\times\omega_2+28\omega_1\times\omega_2)$	$\frac{(\gamma\Delta T)^7 \sin^2 a}{80640}$
3	$\frac{\Delta T}{8}(\omega_0+3\omega_1+3\omega_2+\omega_3)+\frac{(\Delta T)^2}{6720}(29\omega_0\times\omega_3+360\omega_1\times\omega_3+873\omega_2\times\omega_3)$	$\frac{11(\gamma\Delta T)^9 \sin^2 a}{97977600}$
4	$\frac{\Delta T}{90}(7\omega_0+32\omega_1+12\omega_2+32\omega_3+7\omega_4)$ $+\frac{(\Delta T)^2}{56700}(107\omega_0\times\omega_4+1968\omega_1\times\omega_4+3028\omega_2\times\omega_4+6512\omega_3\times\omega_4)$	$\frac{71(\gamma\Delta T)^{11} \sin^2 a}{490497638400}$

Comparing the EDOAs in Table 2 with those in Table 1 indicates that the coning algorithms optimised with angular rates may be superior in the accuracy of attitude computation on condition $\gamma\Delta T < 1$. This conjecture is based on the conventional assumption[6] that the errors of periodic components in the optimal coning algorithms are negligible. However, almost no attention has been paid to examining this negligibility up till now.

4. INFLUENCE OF PERIODIC COMPONENTS

4.1 Change of Rotation Vector

Based on Eqns (5) and (9), the true value of the change of the rotation vector can be deduced as

$$\Delta\tilde{\sigma} = \alpha_s + (\Delta T)^2 \sum_{i=0}^{N-1} M_i(\omega_i \times \omega_N) \tag{12}$$

where

$$\Delta\Phi = \begin{bmatrix}\Delta\Phi_x\\ \Delta\Phi_y\\ \Delta\Phi_z\end{bmatrix} = \begin{bmatrix} -2\sin^2\left(\frac{a}{2}\right)\sin(\gamma\Delta T) \\ -2\sin a \sin\left(\gamma\left(t+\frac{\Delta T}{2}\right)\right)\sin\left(\frac{\gamma\Delta T}{2}\right) \\ 2\sin a\cos\left(\gamma\left(t+\frac{\Delta T}{2}\right)\right)\sin\left(\frac{\gamma\Delta T}{2}\right)\end{bmatrix} \qquad (13)$$

$$\xi = \frac{\frac{\Delta\sigma}{2}}{\sin\left(\frac{\Delta\sigma}{2}\right)} - 1 = \frac{\sin^{-1}\left(\sin\left(\frac{a}{2}\right)\sin\left(\frac{\gamma\Delta T}{2}\right)\right)}{\sin\left(\frac{a}{2}\right)\sin\left(\frac{\gamma\Delta T}{2}\right)\sqrt{1-\left(\sin\left(\frac{a}{2}\right)\sin\left(\frac{\gamma\Delta T}{2}\right)\right)^2}} - 1 \qquad (14)$$

Yan[13], *et al*. took the vector $\Delta\Phi$ denoted by Eqn (13) as the true value of the change of the rotation vector to analyse the residual error along nonperiodic component of Miller's algorithm. This approximation seems necessary for further derivation in this case. However, in the accuracy analysis of attitude computation, the difference between the vectors $\Delta\boldsymbol{\sigma}$ and $\Delta\Phi$ in Eqn (12) (i.e., the coefficient ξ) cannot be ignored.

As described in Eqn (14), the coefficient ξ is a function of the coning frequency $f = \gamma/(2\pi)$ and the coning half-angle a when the updating period ΔT is invariant. This functional relation is illustrated in Fig. 1, where f varies in a wide range from 0.1 Hz to 10 Hz, a changes from 0.1° to 15°, and ΔT is 0.01 s. The surface in Fig. 1 shows that the coefficient ξ increases rapidly, and reaches its maximum value of 1.05098×10^{-3} as the coning frequency and coning half-angle grow. To a great extent, it demonstrates that the coefficient ξ is non-ignorable.

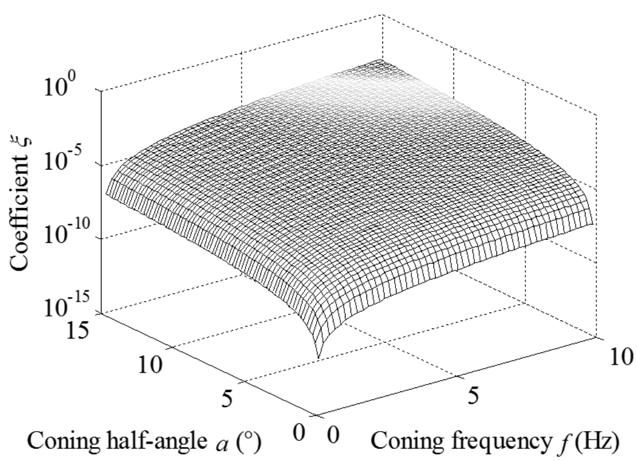

Figure 1. Coefficient ξ (semilog plot in z axis) as function of coning frequency f and coning half-angle a.

4.2 Categorization of Algorithms

Equation (8) is used to express the theoretical limit of the change of the rotation vector under the classical coning motion. As shown in Eqns (8) and (12), there is a difference between the theoretical value and the true value. To clarify this difference, we take advantage of the vector $\Delta\Phi$ denoted by Eqn (13) and define the following parameters

$$\bar{\xi}_x = \frac{\Delta\bar{\sigma}_x}{\Delta\Phi_x} - 1,\quad \bar{\xi}_y = \frac{\Delta\bar{\sigma}_y}{\Delta\Phi_y} - 1,\quad \bar{\xi}_z = \frac{\Delta\bar{\sigma}_z}{\Delta\Phi_z} - 1 \qquad (15)$$

Substituting Eqn (15) into Eqn (8) yields

$$\Delta\bar{\sigma} = \begin{bmatrix}(1+\bar{\xi}_x)\Delta\Phi_x \\ (1+\bar{\xi}_y)\Delta\Phi_y \\ (1+\bar{\xi}_z)\Delta\Phi_z\end{bmatrix} \qquad (16)$$

Equation (16) shows that $\Delta\bar{\sigma}$ can be written in a form similar to Eqn (12). So the deviation of $\Delta\bar{\sigma}$ from the true value $\Delta\boldsymbol{\sigma}$ can be substantiated by comparing the parameters $\bar{\xi}_x$, $\bar{\xi}_y$, and $\bar{\xi}_z$ with the coefficient ξ. For example, in an ideal coning environment where f = 2 Hz, a = 1°, and ΔT = 0.01 s, the approximate results can be obtained that $\bar{\xi}_x = 2.00795 \times 10^{-7}$, $\bar{\xi}_y = \bar{\xi}_z = 2.00478 \times 10^{-7}$, and $\xi = 2.00162 \times 10^{-7}$. It can be seen that the order of magnitude of the deviations along all three axes is 10^{-10}.

This procedure can also be used to analyse the algorithms listed in Tables 1 and 2. For the convenience of further discussion, these algorithms are uniformly denoted by

$$\Delta\tilde{\sigma} = \alpha + \Delta\sigma_s \qquad (17)$$

where

$$\Delta\tilde{\sigma} = \begin{bmatrix}\Delta\tilde{\sigma}_x\\ \Delta\tilde{\sigma}_y\\ \Delta\tilde{\sigma}_z\end{bmatrix},\quad \alpha = \begin{bmatrix}\alpha_x\\ \alpha_y\\ \alpha_z\end{bmatrix},\quad \Delta\sigma_s = \begin{bmatrix}\Delta\sigma_{sx}\\ \Delta\sigma_{sy}\\ \Delta\sigma_{sz}\end{bmatrix} \qquad (18)$$

and $\Delta\boldsymbol{\sigma}_s$ and $\boldsymbol{\alpha}$ are the sum of all cross products and the rest, respectively. Furthermore, five parameters are defined as follows

$$\tilde{\xi}_x = \frac{\Delta\tilde{\sigma}_x}{\Delta\Phi_x} - 1 \qquad (19)$$

$$\eta_y = \frac{\alpha_y}{\Delta\Phi_y} - 1,\quad \eta_z = \frac{\alpha_z}{\Delta\Phi_z} - 1 \qquad (20)$$

$$\mu_y = \frac{\Delta\sigma_{sy}}{\alpha_y},\quad \mu_z = \frac{\Delta\sigma_{sz}}{\alpha_z} \qquad (21)$$

Substituting Eqns (19-21) into Eqn (17), the relationship between $\Delta\tilde{\sigma}$ and $\Delta\Phi$ can be described as

$$\Delta\tilde{\sigma} = \begin{bmatrix}(1+\tilde{\xi}_x)\Delta\Phi_x \\ (1+\eta_y+\mu_y+\eta_y\mu_y)\Delta\Phi_y \\ (1+\eta_z+\mu_z+\eta_z\mu_z)\Delta\Phi_z\end{bmatrix} \qquad (22)$$

So comparing the parameters $\tilde{\xi}_x$, $\eta_y+\mu_y+\eta_y\mu_y$, and $\eta_z+\mu_z+\eta_z\mu_z$ of each algorithm with the coefficient ξ, we can find its deviation from the true value $\Delta\sigma$.

Table 3 shows the parameters $\tilde{\xi}_x$, η_y, η_z, μ_y, and μ_z in the above ideal coning environment. Here, A and B denote the coning algorithms optimised with angular increments and angular rates, respectively.

As shown in Table 3, $\tilde{\xi}_x$ gradually approaches the parameter $\bar{\xi}_x$ calculated above as the number of subintervals increases, for both two types of algorithms. This result proves that the limit of the parameter $\tilde{\xi}_x$ is not ξ but $\bar{\xi}_x$. At the same time, low-magnitude oscillations occur in the parameters μ_y

Table 3. Five parameters in a given environment

	N	$\tilde{\xi}_x$	η_y	η_z	μ_y	μ_z
A	1	2.63675×10^{-3}	0	0	0	0
	2	7.21396×10^{-7}	0	0	2.00491×10^{-7}	2.00491×10^{-7}
	3	2.00834×10^{-7}	0	0	$[1.60434, 2.40534]\times10^{-7}$	$[1.80538, 2.20429]\times10^{-7}$
	4	2.00795×10^{-7}	0	0	$[1.45758, 2.55177]\times10^{-7}$	$[1.73221, 2.27714]\times10^{-7}$
	5	2.00795×10^{-7}	0	0	$[1.37144, 2.63771]\times10^{-7}$	$[1.68926, 2.31989]\times10^{-7}$
B	1	5.05732×10^{-6}	-1.31629×10^{-3}	-1.31629×10^{-3}	2.00689×10^{-7}	2.00689×10^{-7}
	2	2.00893×10^{-7}	8.66266×10^{-8}	8.66266×10^{-8}	$[1.06986, 2.93864]\times10^{-7}$	$[1.53890, 2.46960]\times10^{-7}$
	3	2.00795×10^{-7}	3.84987×10^{-8}	3.84987×10^{-8}	$[1.02443, 2.98372]\times10^{-7}$	$[1.51619, 2.49196]\times10^{-7}$
	4	2.00795×10^{-7}	-2.03559×10^{-12}	-2.03559×10^{-12}	$[1.00950, 2.99853]\times10^{-7}$	$[1.50873, 2.49930]\times10^{-7}$

and μ_z, and the absolute values of the parameters η_y and η_z of the second type decline rapidly. Note that the deviations of the coning algorithms along y and z axes are determined by these four parameters.

Taking the limit denoted by Eqn (8) into account, the algorithms involved in Table 3 can be categorized into four groups as compared with the true value $\Delta\sigma$

(1) $\tilde{\xi}_x - \xi \gg (\eta_y + \mu_y + \eta_y\mu_y) - \xi$ and $\tilde{\xi}_x - \xi \gg (\eta_z + \mu_z + \eta_z\mu_z) - \xi$. The 1-subinterval and 2-subinterval algorithms optimised with angular increments are ascribed to this group. Accordingly, Eqn (22) can be simplified as

$$\Delta\tilde{\sigma} = \begin{bmatrix} (1+\tilde{\xi}_x)\Delta\Phi_x \\ (1+\tilde{\xi}_y)\Delta\Phi_y \\ (1+\tilde{\xi}_z)\Delta\Phi_z \end{bmatrix} = \begin{bmatrix} \Delta\tilde{\sigma}_x \\ \Delta\bar{\sigma}_y \\ \Delta\bar{\sigma}_z \end{bmatrix} \tag{23}$$

(2) $\eta_y \gg \mu_y$ and $\eta_z \gg \mu_z$. This refers to the 1-subinterval algorithm optimised with angular rates. The corresponding approximation of Eqn (22) is

$$\Delta\tilde{\sigma} = \begin{bmatrix} (1+\tilde{\xi}_x)\Delta\Phi_x \\ (1+\eta_y)\Delta\Phi_y \\ (1+\eta_z)\Delta\Phi_z \end{bmatrix} = \begin{bmatrix} \Delta\bar{\sigma}_x \\ (1+\eta_y)\Delta\Phi_y \\ (1+\eta_z)\Delta\Phi_z \end{bmatrix} \tag{24}$$

(3) The parameters η_y and μ_y are not negligible as compared with η_z and μ_z, respectively. The 2-subinterval and 3-subinterval algorithms optimised with angular rates are included in this group, where the simplified form of Eqn (22) can be expressed

$$\Delta\tilde{\sigma} = \begin{bmatrix} (1+\tilde{\xi}_x)\Delta\Phi_x \\ (1+\eta_y+\mu_y)\Delta\Phi_y \\ (1+\eta_z+\mu_z)\Delta\Phi_z \end{bmatrix} = \begin{bmatrix} \Delta\bar{\sigma}_x \\ (1+\eta_y+\mu_y)\Delta\Phi_y \\ (1+\eta_z+\mu_z)\Delta\Phi_z \end{bmatrix} \tag{25}$$

(4) $\eta_y \ll \mu_y$ and $\eta_z \ll \mu_z$. This group is composed of the 3-subinterval, 4-subinterval, and 5-subinterval algorithms optimised with angular increments, and the 4-subinterval algorithm optimised with angular rates. In this case, the approximated expression for Eqn (22) is

$$\Delta\tilde{\sigma} = \begin{bmatrix} (1+\bar{\xi}_x)\Delta\Phi_x \\ (1+\mu_y)\Delta\Phi_y \\ (1+\mu_z)\Delta\Phi_z \end{bmatrix} = \begin{bmatrix} \Delta\bar{\sigma}_x \\ (1+\mu_y)\Delta\Phi_y \\ (1+\mu_z)\Delta\Phi_z \end{bmatrix} \tag{26}$$

As shown in Eqns (24-26), the difference between the optimal coning algorithm and the theoretical value $\Delta\bar{\sigma}$ is not revealed by its component on x axis any longer. In other words, the influence of periodic components in these algorithms is dominant. It should be noted that all algorithms optimised with angular rates are included in the last three groups.

5. SIMULATIONS

To validate the influence of periodic components, a variety of pitch-error simulations are carried out based on the two types of algorithms and their simplified forms denoted by Eqns (23-26).

In the simulation, the pitch error is defined as $\delta\theta = |\theta - \tilde{\theta}|$, where θ and $\tilde{\theta}$ are the pitches calculated from Eqns (2) and (4) at the same time, respectively. Table 4 shows the ideal coning environments for simulation. The updating period of attitude quaternion is 0.01 s, and the duration of simulation is 36 s.

The numerical results in the coning environment 3# are illustrated in Figs. 2 and 3. For brevity's sake, we omit the results in the other coning environments because of their high degree of similarity to those in the environment 3#.

In Fig. 2, the pitch errors based on the simplified forms (SF) are close to those based on the coning algorithms optimised with angular increments (CAOWAI), except for the case $N = 3$. This discrepancy is related to the corresponding parameter $\tilde{\xi}_x$

Table 4. Coning environments for simulation.

Environment	Coning frequency (Hz)	Coning half-angle (deg)
1#	0.2	0.1
2#	0.2	1
3#	2	1
4#	2	10
5#	20	10

in Table 3, which reveals that the optimisation of nonperiodic component in the 3-subinterval algorithm is inadequate. In contrast, the curves in Fig. 3 show good agreement between the pitch errors based on the coning algorithms optimised with angular rates (CAOWAR) and their simplified forms (SF).

Additionally, Fig. 2 reveals that when N exceeds 3 the CAOWAIs do not have substantial improvement in pitch accuracy any longer. Thus the 4-subinterval and 5-subinterval algorithms are not recommended on the consideration of their complex structures (see Table 1). On the other hand, comparing with Fig. 2, the CAOWAR with identical N is not obviously superior in pitch accuracy. This is not consistent with the

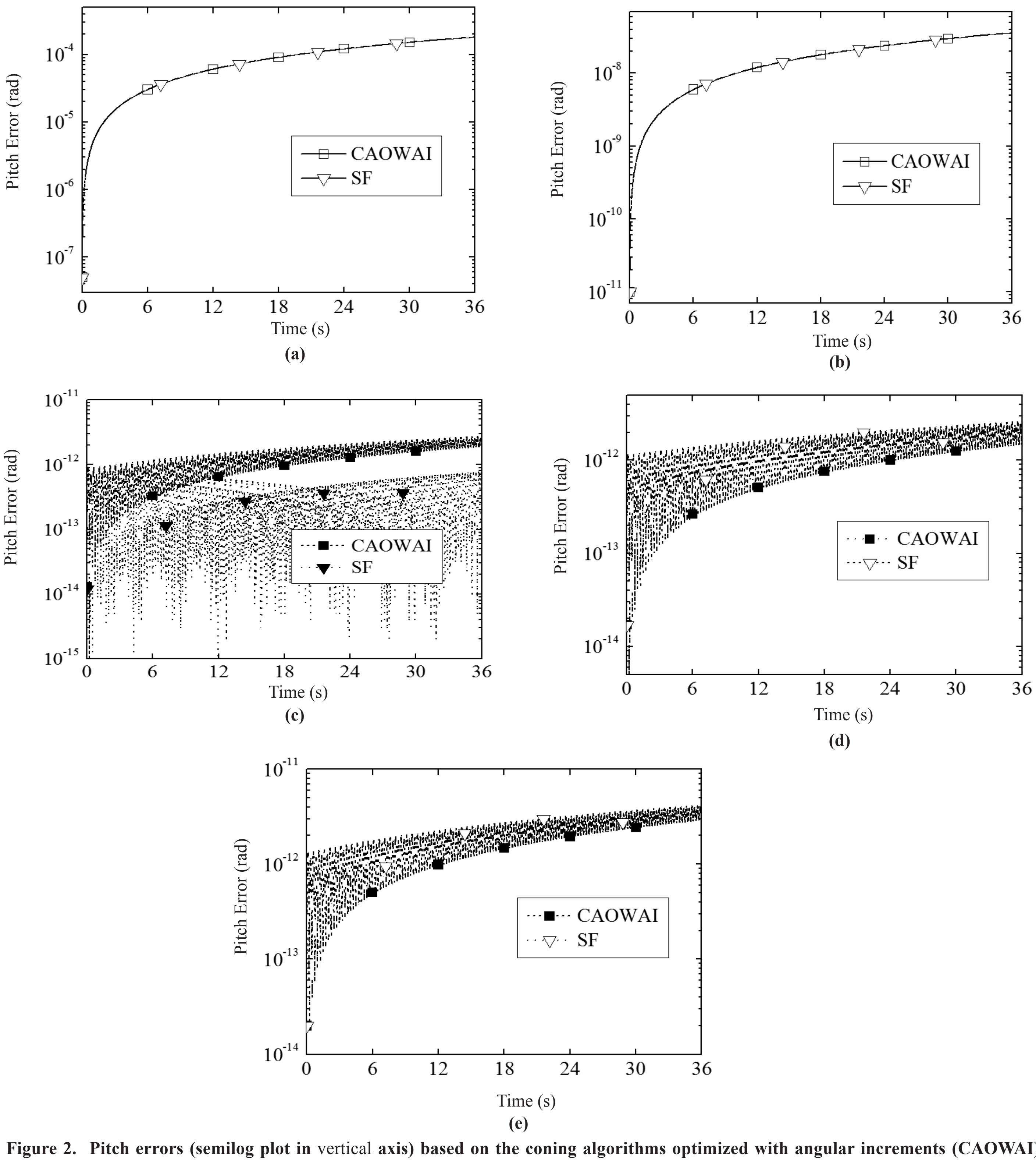

Figure 2. Pitch errors (semilog plot in vertical **axis) based on the coning algorithms optimized with angular increments (CAOWAI) and their simplified forms (SF) for: (a) N = 1, b) N = 2, (c) N = 3, (d) N = 4, and (e) N = 5.**

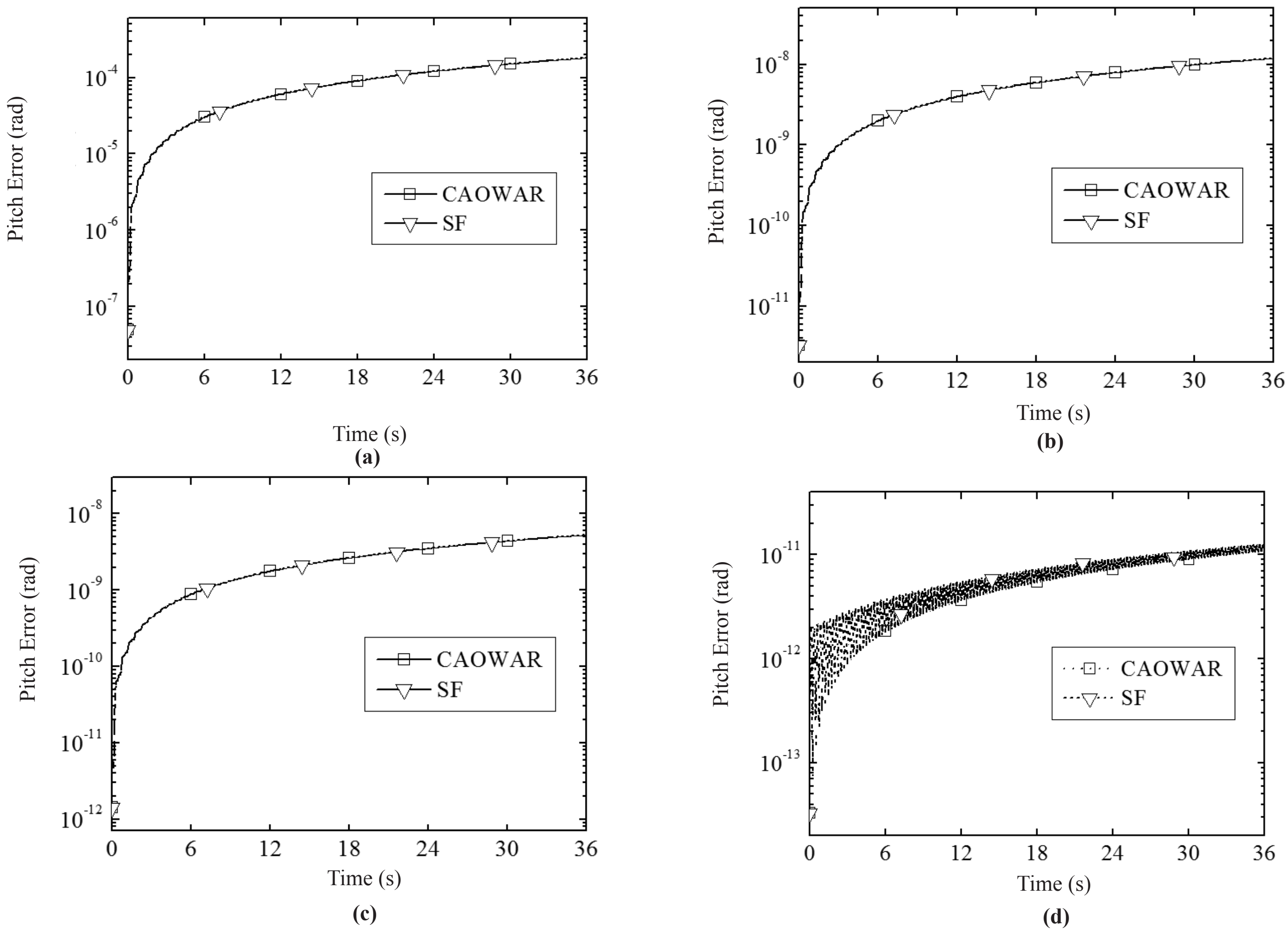

Figure 3. Pitch errors (semilog plot in vertical axis) based on the coning algorithms optimised with angular rates (CAOWAR) and their simplified forms (SF) for: (a) *N* = 1, (b) *N* = 2, (c) *N* = 3, and (d) *N* = 4.

conjecture stated at the end of section 3. The explanation for this phenomenon is that the periodic components in CAOWARs are dominant factors, just as what has been discussed earlier. So it is necessary to further optimise these periodic components.

6. CONCLUSION

This study analyses the accuracy of attitude computation based on two types of optimal coning algorithms under the classical coning motion. After deriving the true value of the change of the rotation vector, we explore the influence of periodic components in these algorithms using analytical comparison and categorization. Analytical results indicate that the influence of periodic components is dominant in these algorithms except for the 1-subinterval and 2-subinterval ones optimised with angular increments. Moreover, numerical tests are constructed, and the results agree well with the analyses. Allowing for the comparison of accuracy, the 4-subinterval and 5-subinterval algorithms optimised with angular increments are not recommended for use. To improve the accuracy of attitude computation, future work will concentrate on the optimisation of periodic components in existing coning algorithms.

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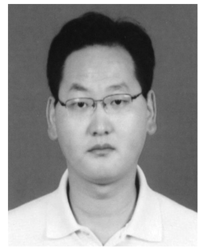

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