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Approximate Solution and its Convergence Analysis for Hypersingular Integral Equations

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ABSTRACT

This paper proposes a residual based Galerkin method with Legendre polynomials as basis functions to find the approximate solutions of hypersingular integral equations. These equations occur quite naturally in the field of aeronautics such as problem of aerodynamics of flight vehicles and during mathematical modelling of vortex wakes behind an aircraft. The analytic solution of these kind equations is known only for a particular case when $m(x,s)=0$ *in* $\int_{-1}^{1}\frac{\xi(s)}{(s-x)^2}ds-\int_{-1}^{1}m(x,s)\xi(s)ds=z(x)$. Also, in these singular integral equations which occur during the formulation of many boundary value problems, the known function $m(x,s)$, is not always zero. Our proposed method finds the approximate solutions by converting the integral equations into a linear system of algebraic equations which is easy to solve. The convergence of sequence of approximate solutions is proved and error bound is obtained theoretically. The validation of derived theoretical results and implementation of method is also shown with the aid of numerical illustrations.

Keywords: Hypersingular kernel; Galerkin method; Convergence analysis; Condition number

1. INTRODUCTION

Hypersingular integral equations play a vital role in the field of aeronautics[1-3]. These integral equations occur during the formulation of interference or interaction problems such as wing and tail surfaces problem, pairs or collections of wings (biplanes or cascades) problems[2] and mathematical modelling of vortex wakes at the time of takeoff-landing operations[4]. Also, in study of theory of incompressible flow we face the problem of evaluating the hypersingular integral like Prandtl's integral equation formulated as singular integral equation to calculate the circulation distribution[5,6] of a finite span wing.

Apart from aeronautics, the problems of acoustics[7], fluid dynamics[8], fracture mechanics[9] and water wave scattering[10] can be modelled as hypersingular integral equations. Many analytical and numerical methods such as polynomial approximation method[9], complex variable function method[11] and reproducing kernel method[12] for solving singular integral equations have been already explored. However, search for a method which is easy to understand, easy to implement and numerically stable is always there. In this article, we propose a residual based Galerkin method to find the approximate solutions of integral equations with hyper kernel.

The hypersingular integral equations that occur during the formulation of many boundary value problems of practical interest are of the form

$$\int_{-1}^{1}\frac{\xi(s)}{(s-x)^2}ds-\int_{-1}^{1}m(x,s)\xi(s)ds=z(x),\ |x|<1 \tag{1}$$

with $\xi(\pm1)=0$, *where* $z(x)$, $m(x,s)$ are known real valued H¨older continuous[13] functions and $\xi(x)$ is an unknown function. The function $\xi(x)$ is assumed to have the H¨older continuous derivative of first order on $(-1,1)$ which is required to ensure the existence of finite-part integral[14]. In Eqn. (1), the singular integral exists as Hadamard finite part integral (HFP) which is defined as

$$\int_{-1}^{1}\frac{\xi(s)}{(s-x)^2}ds=\lim_{\epsilon\to0^+}\left[\int_{-1}^{x-\epsilon}\frac{\xi(s)}{(s-x)^2}ds+\int_{x+\epsilon}^{1}\frac{\xi(s)}{(s-x)^2}ds-\frac{\xi(x+\epsilon)+\xi(x-\epsilon)}{\epsilon}\right],\ |x|<1 \tag{2}$$

2. METHOD OF SOLUTION

A function $\xi(s)$ defined on $[-1,1]$ in Eqn. (1) with $\xi(\pm1)=0$ can be represented as follows

$$\xi(s)=\sqrt{1-s^2}\,\phi(s) \tag{3}$$

where $\phi(s)$ is an unknown function defined on $[-1,1]$. Using Eqn. (3) in Eqn. (1), we obtain

$$\int_{-1}^{1}\frac{\phi(s)\sqrt{1-s^2}}{(s-x)^2}ds-\int_{-1}^{1}m(x,s)\phi(s)\sqrt{1-s^2}\,ds=z(x),\ |x|<1 \tag{4}$$

Now we approximate the function $\phi(s)$ as follows

$$\phi(s)\approx\phi_n(s)=\sum_{j=0}^{n}\alpha_j e_j(s) \tag{5}$$

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where $\{e_j(s)\}_{j=0}^{n}$ denotes the set of $(n+1)$ orthonormalised Legendre polynomials on $[-1,1]$ and $\alpha_j;\, j=1,2,\ldots,n,$ are unknown constant coefficients. On using the approximation which is defined in Eqn. (5) for $\phi(s)$ in Eqn. (4), the residual error $R(x,\alpha_0,\alpha_1,\alpha_2,...,\alpha_n)$ is as follows

$$R(x,\alpha_0,\alpha_1,\alpha_2,...,\alpha_n)=\int_{-1}^{1}\frac{\phi_n(s)\sqrt{1-s^2}}{(s-x)^2}ds-\int_{-1}^{1}m(x,s)\phi_n(s)\sqrt{1-s^2}\,ds-z(x),\ |x|<1 \tag{6}$$

In Galerkin method, this $R(x,\alpha_0,\alpha_1,\alpha_2,...,\alpha_n)$ is assumed to be orthogonal to the space spanned by orthonormal polynomials, say $E=span\{e_j(x)\}_{j=0}^{n}$ that is, we have

$$\langle R(x,\alpha_0,\alpha_1,\alpha_2,...,\alpha_n),e_j\rangle_{L^2}=0,\forall j=0,1,2,\ldots,n \tag{7}$$

where L^2 is the space of all real valued functions which are square integrable on $[-1,1]$.

Using Eqn. (6) for $j=0,1,2,\ldots,n,$ Eqn. (7) becomes

$$\left\langle\int_{-1}^{1}\frac{\phi_n(s)\sqrt{1-s^2}}{(s-x)^2}ds-\int_{-1}^{1}m(x,s)\phi_n(s)\sqrt{1-s^2}\,ds-z(x),e_j\right\rangle_{L^2}=0 \tag{8}$$

We evaluate singular integral for $j=0,1,\ldots,n$, in system (8) by using results[15] (see Eqn. (34)[15]) and we get a linear system of order $(n+1)\times(n+1)$. The matrix form of the above system (8) is as follows

$$C_n^T A_n=\hat{C}_n A_n=\hat{Z}_n \tag{9}$$

where

$$\hat{C}_n=C_n^T,\ C_n=\begin{pmatrix}c_{00}&\cdots&c_{0n}\\ \vdots&\ddots&\vdots\\ c_{n0}&\cdots&c_{nn}\end{pmatrix},\quad A_n=\begin{pmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_n\end{pmatrix},\quad \hat{Z}_n=\begin{pmatrix}z_0\\ z_1\\ \vdots\\ z_n\end{pmatrix},$$

$$c_{rq}=\int_{-1}^{1}\left(\int_{-1}^{1}\frac{\sqrt{1-s^2}\,e_r(s)}{(s-x)^2}ds-\int_{-1}^{1}m(x,s)e_r(s)\sqrt{1-s^2}\,ds\right)e_q(x)dx,\ r,\ q=0,1,2,\ldots,n,$$

$$z_q=\int_{-1}^{1}z(x)e_q(x)dx,\ q=0,1,2,\ldots,n. \tag{10}$$

We evaluate singular integral for $j=0,1,\ldots,n$, in Eqn. (8) by using results[15] (see formula (34) of ref[15]) and we get a linear system of order $(n+1)\times(n+1)$.

Solving Eqn. (9), we obtain the value of unknown coefficients $\alpha_j;\, j=0,1,2,....,n.$ the substitution of these α_j values in Eqn. (5) provides the approximate solution of Eqn. (4) and hence for Eqn. (1). This completes the description of proposed method used to find an approximate solution of Eqn. (1).

3. CONVERGENCE ANALYSIS

In this section, we show that sequence $\{\phi_n\}_{n=0}^{\infty}$ converges to the exact solution in L^2 space and we derive the error bound.

3.1 Function Spaces

We initialize this subsection by defining function spaces in which the error analysis of numerical method takes place.

$L^2[-1,1]=\{\mathcal{U}(s):[-1,1]\to\mathbb{R}:\int_{-1}^{1}(\mathcal{U}(s))^2ds<\infty\}$ is a Hilbert Space of all square integrable real valued functions defined on $[-1,1]$, equipped with the norm $\|\cdot\|_{L^2}$ and inner product $\langle .,.\rangle_{L^2}$, defined as

$$\|\mathcal{U}(s)\|_{L^2}=\left(\int_{-1}^{1}(\mathcal{U}(s))^2ds\right)^{1/2}\ for\ \mathcal{U}(s)\in L^2[-1,1] \tag{11}$$

$$\langle\mathcal{U}_1,\mathcal{U}_2\rangle_{L^2}=\int_{-1}^{1}\mathcal{U}_1(s)\mathcal{U}_2(s)ds\ for\ \mathcal{U}_1,\ \mathcal{U}_2\in L^2[-1,1] \tag{12}$$

Now using the concept[16], we define the set of functions

$$V=\{\mathcal{U}(s)\in L^2:\sum_{j=0}^{\infty}d_j^{\,2}\langle\mathcal{U},e_j\rangle_{L^2}^2<\infty\} \tag{13}$$

$$d_j^{\,2}=\|Se_j\|_{L^2}^2 \tag{14}$$

$$Se_j(x)=\int_{-1}^{1}\frac{\sqrt{1-s^2}}{(s-x)^2}e_j(s)ds \tag{15}$$

The set V is a subspace of L^2 space which is made into a Hilbert space with the following norm $\|\cdot\|_V$ and inner product $\langle .,.\rangle_V$

$$\|\mathcal{U}\|_V^2=\sum_{j=0}^{\infty}d_j^{\,2}\langle\mathcal{U},e_j\rangle_{L^2}^2\ for\ \mathcal{U}(s)\in V \tag{16}$$

$$\langle\mathcal{U}_1,\mathcal{U}_2\rangle_V=\sum_{j=0}^{\infty}d_j^{\,2}\langle\mathcal{U}_1,e_j\rangle_{L^2}\langle\mathcal{U}_2,e_j\rangle_{L^2}\ \ for\ \ \mathcal{U}_1(s),\ \mathcal{U}_2(s)\in V \tag{17}$$

where d_j is same as defined in Eqn. (14). Let $h_k=\frac{e_k}{d_k}$, then $\|h_k\|_V=1$. This set $\{h_k\}_{k=0}^{\infty}$ forms complete orthonormal basis for the Hilbert space V, that is if $\mathcal{U}\in V$, then we have

$$\mathcal{U}(s)=\sum_{k=0}^{\infty}\langle\mathcal{U},h_k\rangle_V h_k(s) \tag{18}$$

Now operating the operator S which is defined in Eqn. (15) on orthonormalised Legendre polynomials $e_j(x);\, j=0,1,2,\ldots,n$ and using the results[15] (see Eqn. (34)[15]), we obtain

$$Se_n(s)=\sum_{i=0}^{n}\beta_i e_i(s);\ where\ \ \beta_i=\langle Se_n,e_i\rangle_{L^2},i=0,1,2,\ldots,n \tag{19}$$

3.2 Error Bound

With the help of Eqn. (19), we can extend the operator $S:V\to L^2$ as a bounded linear operator and defined as

$$S\phi(x)=\sum_{j=0}^{\infty}\langle\phi,e_j\rangle_{L^2}\sum_{i=0}^{j}\langle Se_j,e_i\rangle_{L^2}e_i(x)\in L^2[-1,1] \tag{20}$$

Using Eqn. (20), we find the norm of bounded linear operator S

$$\|S\phi\|_{L^2}^2=\sum_{j=0}^{\infty}d_j^2\langle\phi,e_j\rangle_{L^2}^2=\|\phi\|_V^2 \tag{21}$$

Hence using Eqn. (21), we obtain

$$\|S\|=1 \tag{22}$$

Moreover, the mapping $S:V\to L^2$ is one-one and onto. Therefore following Bounded Inverse Theorem[17], the operator

$S^{-1}:L^2\to V$ exists as a bounded linear operator and which is defined as

$$S^{-1}\phi(x)=\sum_{j=0}^{\infty}\frac{\langle\phi(x),e_j(x)\rangle_{L^2}}{d_j}e_j(x) \tag{23}$$

Now, with the aid of Eqn. (23), we calculate the norm for linear operator S^{-1}

$$\|S^{-1}\phi(x)\|_V=\|\phi(x)\|_{L^2} \tag{24}$$

Finally, using Eqn. (24), the norm of bounded operator S^{-1} is

$$\|S^{-1}\|=1 \tag{25}$$

Now we consider the mapping $Q_n:L^2\to L^2$, where Q_n is an orthogonal projection operator which is defined as

$$Q_n\phi(x)=\sum_{j=0}^{n}\langle\phi,e_j\rangle_{L^2}e_j(x) \tag{26}$$

where n is the degree of orthonormalised Legendre polynomial by which $\phi(x)$ is approximated. Now we write (4) in an operator equation from the spaces V to L^2

$$(S-M)\phi(x)=z(x),\ z(x)\in L^2,\ \phi(x)\in V \tag{27}$$

where the operator S is defined in Eqn. (15) and we define the operator $M:V\to L^2$ as follows

$$M\phi(x)=\int_{-1}^{1}m(x,s)\phi(s)\sqrt{1-s^2}\,ds \tag{28}$$

The operator $M:V\to L^2$, defined in Eqn. (28) will be a compact operator with the following assumption

$$\int_{-1}^{1}\int_{-1}^{1}m_1^2(x,s)\,ds\,dx<\infty,\quad m_1=m(x,s)\sqrt{1-s^2} \tag{29}$$

Equation (27) will be having a unique solution if and only if the inverse of the operator $(S-M)$ exists as a bounded linear operator. We assume that the bounded linear operator $(S-M)^{-1}$ exists. From Eqn. (7), we have

$$Q_n[(S-M)\phi_n(x)-z(x)]=0 \tag{30}$$

Since the function $S\phi_n(x)$ is a polynomial therefore following the definition of operator Q_n, we get

$$Q_nS\phi_n(x)=S\phi_n(x) \tag{31}$$

Using the above fact, Eqn. (30) becomes

$$S\phi_n(x)-Q_nM\phi_n(x)=Q_nz(x) \tag{32}$$

Since the operator S has a bounded inverse and the operator M is compact, hence for all $n>n_0$, $(S-Q_nM)^{-1}$ exists as a bounded linear operator[13]. Therefore, Eqn. (32) has a unique solution, which is as follows

$$\phi_n(x)=(S-Q_nM)^{-1}Q_nz(x) \tag{33}$$

Now from Eqns. (27) and (33), we have

$$\phi(x)-\phi_n(x)=(S-Q_nM)^{-1}\big(z(x)-Q_nz(x)+M\phi(x)-Q_nM\phi(x)\big) \tag{34}$$

Taking norm of both the sides of Eqn. (34), we obtain

$$\|\phi-\phi_n\|_V\le\|(S-Q_nM)^{-1}\|\left(\|z-Q_nz\|_{L^2}+\|M\phi(x)-Q_nM\phi(x)\|_{L^2}\right) \tag{35}$$

Due to the assumption defined in Eqn. (29), the operator M is a Hilbert-Schmidt operator[13] and hence $\|M-Q_nM\|_{L^2}\to 0$ as $n\to\infty$. Also, we have $\|z-Q_nz\|_{L^2}\to 0$ as $n\to\infty$. Thus, we get $\|\varphi-\varphi_n\|_V\to 0$ as $n\to\infty$. Further, due to the fact that if $\phi\in V$ then, we have

$$\|\phi\|_{L^2}\le\|\phi\|_V \tag{36}$$

On using Eqn. (36), Eqn. (35) can be written as follows

$$\|\phi-\phi_n\|_{L^2}\le\|(S-Q_nM)^{-1}\|\left(\|z-Q_nz\|_{L^2}+\|M\phi(x)-Q_nM\phi(x)\|_{L^2}\right) \tag{37}$$

Hence, sequence $\{\phi_n\}_{n=0}^{\infty}$ converges to the exact solution in L^2 space.

3.3 Theorem

If $cond(\hat{C}_n)$ denotes the condition number of coefficient matrix $\hat{C}_n$, where $cond(\hat{C}_n)=\|\hat{C}_n\|\|\hat{C}_n^{-1}\|$ and the norm of matrices are the spectral norm. Then, we have

$$\lim_{n\to\infty}cond(\hat{C}_n)=cond(S-M) \tag{38}$$

Proof: We define prolongation operator[13], $\mathcal{P}_n:\mathbb{R}^{n+1}\to E$ as

$$\mathcal{P}_n\hat{Z}_n=\sum_{j=0}^{n}\langle z,e_j\rangle_{L^2}e_j(s)\in E \tag{39}$$

where $\mathbb{R}^{n+1}$ is a real vector space having $(n+1)$-tuples of real numbers. Also, we have

$$Q_nz(s)=\sum_{j=0}^{n}\langle z,e_j\rangle_{L^2}e_j(s) \tag{40}$$

Following Eqns. (39) and (40), we get

$$\mathcal{P}_n\hat{Z}_n=Q_nz(s),\ |s|<1 \tag{41}$$

Further, we define a restriction operator[13] $\mathcal{R}_n:E\to\mathbb{R}^{n+1}$ as

$$\mathcal{R}_n\phi_n(s)=(\langle\phi_n,e_0\rangle_{L^2},\langle\phi_n,e_1\rangle_{L^2},\ldots\langle\phi_n,e_n\rangle_{L^2})^T\in\mathbb{R}^{n+1} \tag{42}$$

By orthogonality of Legendre polynomials in Eqn. (5), we get

$$\alpha_j=\langle\phi_n,e_j\rangle_{L^2},\ j=0,1,\ldots,n \tag{43}$$

From Eqns. (42) and (43), we obtain

$$\mathcal{R}_n\phi_n(s)=A \tag{44}$$

where A is defined in Eqn. (10). The existence of the operator $(S-Q_nM)^{-1}$ implies that $\phi_n(s)$ exists uniquely. Therefore, by Eqn. (44) it is clear that matrix A exists uniquely which proves that the system (9) has a unique solution.

Now we have

$$\hat{C}^{-1}\hat{Z}_n=\mathcal{R}_n(S-Q_nM)^{-1}\mathcal{P}_n\hat{Z}_n \tag{45}$$

$$\|\hat{C}^{-1}\|\le\|\mathcal{R}_n\|\ \|(S-Q_nM)^{-1}\|\ \|\mathcal{P}_n\|=\|(S-Q_nM)^{-1}\| \tag{46}$$

Also, using the definitions of prolongation operator $\mathcal{P}_n$ and the restriction operator $\mathcal{R}_n$, we have

$$\hat{C}=\mathcal{R}_n(S-Q_nM)\mathcal{P}_n \tag{47}$$

$$\|\hat{C}\|\le\|\mathcal{R}_n\|\ \|(S-Q_nM)\|\ \|\mathcal{P}_n\|=\|(S-Q_nM)\| \tag{48}$$

Using Eqns. (46) and (48), we have

$$\|\hat{C}\|\ \|\hat{C}^{-1}\|\le\|(S-Q_nM)\|\ \|(S-Q_nM)^{-1}\| \tag{49}$$

The boundedness of $(S-Q_nM)$ and $(S-Q_nM)^{-1}$ implies condition number of $\hat{C}$ is also bounded. Hence, our proposed method is numerically stable[13]. Since $\|M-Q_nM\| \to 0$, we obtain

$$\begin{aligned} &\|(S-M)-(S-Q_nM)\| \le \|S-M\| + \|S-Q_nM\| \\ &\Rightarrow \|S-Q_nM\| \to \|S-M\| \; as \, n \to \infty \end{aligned} \tag{50}$$

and

$$\begin{aligned} &\|(S-M)^{-1}-(S-Q_nM)^{-1}\| = \|(S-M)^{-1}\left(I-(S-M)(S-Q_nM)^{-1}\right)\| \\ &\le \|(S-M)^{-1}\| \;\; \|(S-Q_nM)^{-1}\| \;\; \|M-Q_nM\| \end{aligned} \tag{51}$$

where I is an identity operator. From Eqn. (51), we obtain

$$\|(S-Q_nM)^{-1}\| \to \|(S-M)^{-1}\| \; as \, n \to \infty \tag{52}$$

Now, from Eqns. (49), (50) and (51), we have

$$\|\hat{C}\| \;\; \|\hat{C}^{-1}\| \to \|S-M\| \;\; \|(S-M)^{-1}\| \; as \, n \to \infty \tag{53}$$

4. EXAMPLES

All the numerical calculations are performed on Wolfram Mathematica 11.0.

Example 1. The hypersingular integral equation

$$\int_{-1}^{1} \frac{\phi(s)\sqrt{1-s^2}}{(s-x)^2} ds + \frac{1}{18}\int_{-1}^{1} (s+x)\phi(s)\sqrt{1-s^2}\, ds = \pi\left(\frac{1}{2}+\frac{x}{144}-3x^2\right), \;\; |x|<1 \tag{54}$$

has exact solution $\phi(x)=x^2$.

It can be seen from Table 1 that we are getting the exact solution at just $n = 2$. Table 1 contains all the numerical results for Example 1. The comparison of exact and approximate solutions for $n = 1, 2$ is shown in Figure 1. And, it is clear from the figure that the exact solution coincides with the approximate solution even for a small value of n which is in this case is $n = 2$.

Table 1. Details of obtained numerical results for different n for Example 1

n	Actual Error (In L^2 norm)	Error bound
1	0.483001	4.25580
2	0	0

Example 2. Consider hypersingular integral equation

$$\int_{-1}^{1} \frac{\phi(s)\sqrt{1-s^2}}{(s-x)^2} ds + \int_{-1}^{1} sx\phi(s)\sqrt{1-s^2}\, ds = \pi\left(-8x^3+\frac{17}{8}x-1\right), \; |x|<1 \tag{55}$$

For which $\phi(x)=1+2x^3$ is the exact solution.

Table 2. Details of obtained numerical results for different n for Example 2

n	Actual Error (In L^2 norm)	Error bound
1	0.46595	9.29106
2	0.46595	9.29106
3	0	0

Again, it can be seen from Table 2 that the approximate solution obtained is identical to the exact solution at just $n = 3$. Although, Chen[12] also solved this problem up to $n = 25$ by using method of reproducing kernel, but his method did

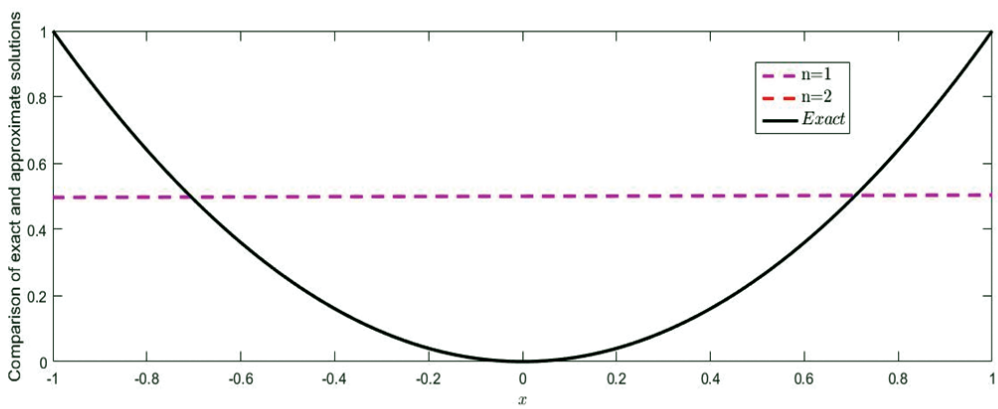

Figure 1. Comparison of exact solution with approximate solutions of Example 1.

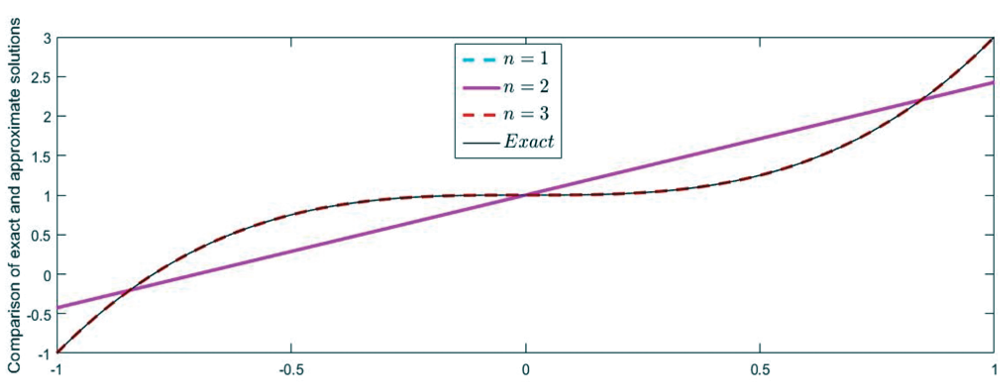

Figure 2. Comparison of exact solution with approximate solutions of Example 2.

not give the exact solution. Figure 2 shows the comparison between approximate solutions and exact solution for different values of n.

Further, it can be seen from the figure that both the solutions coincide. All numerical results are detailed in Table 2.

Example 3. Consider a hypersingular integral equation

$$\int_{-1}^{1}\frac{\phi(s)\sqrt{1-s^2}}{(s-x)^2}ds+\int_{-1}^{1}\frac{(x+x^2)\phi(s)\sqrt{1-s^2}}{36+12s}ds=\pi z(x),\ |x|<1 \quad (56)$$

where

$$z(x)=\frac{1326099}{655360}-\frac{1469711672063x}{7864320}+\frac{84573531x}{320\sqrt{2}}-\frac{1470155415887x^2}{7864320}+\frac{84573531x^2}{320\sqrt{2}}+\frac{115527x^3}{10240}+\frac{4953727x^4}{16384}-\frac{88851x^5}{2560}-\frac{5394557x^6}{10240}+\frac{7571x^7}{320}+\frac{1453239x^8}{5120}+\frac{327x^9}{64}-\frac{1793x^{10}}{256}-\frac{45x^{11}}{16}+\frac{1885x^{12}}{128}$$

The exact solution of this example is

$$\phi(x)=\frac{1}{640}\left(-252+45x+4510x^2-725x^3-22258x^4+2680x^5+38000x^6-2000x^7-20252x^8-252x^9+45x^{10}+150x^{11}-725x^{12}\right).$$

Table 3 contains all the obtained numerical results for Example 3. The comparison between approximate solutions and exact solution for n = 1, 2, . . . , 12, is shown in Figure 3. This figure shows, that the exact solution is in great agreement with the approximate solution at n = 12. It is also clear from Table 3 that the actual error is lying with in the error bound which follows from our result defined by Eqn. (37).

5. CONCLUSIONS

Our numerical method finds approximate solutions of hypersingular integral equations by converting it into a linear system of algebraic equations. The convergence of sequence of approximate solutions is proved in L^2 space and error bound is also derived. The existence and uniqueness for the solution of linear system which is obtained as a result of approximation of Eqn. (1), is also shown.

Table 3. Details of obtained numerical results for different n for Example 3

n	Actual Error (In L^2 norm)	Error bound
1	0.38064	8.84521
2	0.37161	8.74921
3	0.37160	8.74689
4	0.28343	7.76302
5	0.27503	7.64867
6	0.25688	7.24213
7	0.25274	7.15094
8	0.00570	0.19899
9	0.00564	0.19739
10	0.00054	0.02196
11	0.00049	0.02053
12	5.95198×10^{-16}	1.07677×10^{-10}

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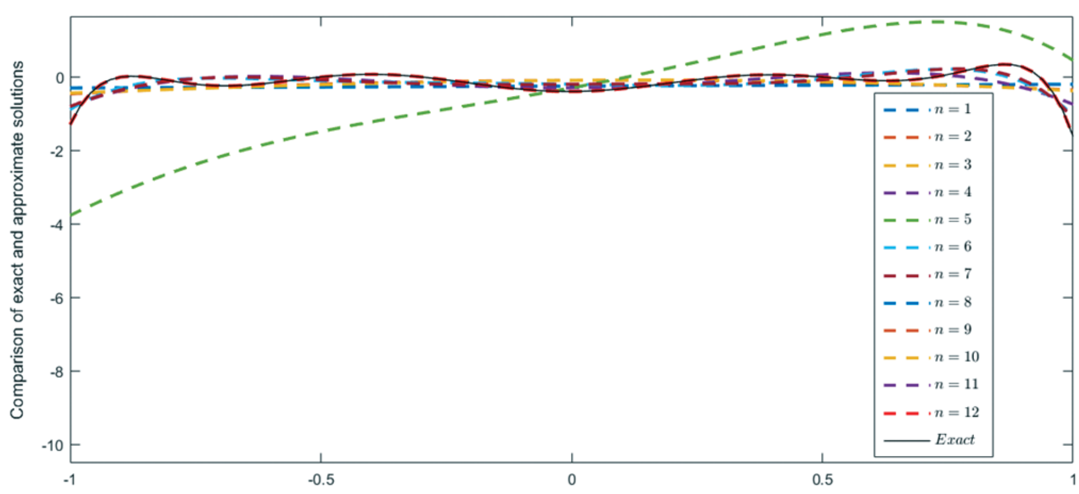

Figure 3. Comparison of exact solution with approximate solutions of Example 3.

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