A Generic Method for Azimuthal Map Projection

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ABSTRACT

Map projections are mathematical methods for projecting spherical coordinates in the form of \((\phi, \lambda)\) to the map coordinates in the form of \((X, Y)\) in Cartesian reference frame. Numerous methods for map projection have been derived and are being used for preparation of cartographic products. These map projections take into account specific position of the viewer on the datum surface for derivation of the map projections. A generic method for azimuthal map projection where the position of the viewer can be taken at an arbitrary point on the datum surface is derived. Using this generic method all the specific azimuthal map projections can be derived.

Keywords: Map projection, azimuthal projection, cartographic transformation, GIS

1. INTRODUCTION

Map projections are a class of cartographic transformation which is important for preparation of maps in different forms. Mathematically map projection is a pair of transformation equations which intakes the locations of objects on 3-D earth surface described through a geodetic datum and transforms them to locations into a 2-D map container. In other words map projection is a pair of 3-D to 2-D transformation.

Another definition of map projection can be transformation of spherical co-ordinate system (in the form of latitude longitude of a datum surface) into Cartesian co-ordinate system on map container. The map container can be a paper map, a digital terrain elevation model, a digital surface model or a vector or raster digital map. Mathematically it can be described as pair of Eqs:\n
\[X = f_1(\phi, \lambda)\]
\[Y = f_2(\phi, \lambda)\]

The reverse process of locating objects on the datum surface given its corresponding location in the map is known as the reverse map projection or inverse map projection. Reverse map projections are described by the following Eqs:

\[\phi = f_1^{-1}(X, Y)\]
\[\lambda = f_2^{-1}(X, Y)\]

Also it can be observed that map projections is a mixture of many-to-one, one-to-one and one-to-many transformation process, where many locations of the spherical datum surface are projected on the map surface.

Also it can be inferred that map projection inherently depends upon the geodetic datum which models the natural surface of the map. There are many theoretical possibilities of map projection. Generally map projections are derived depending upon the datum, viewer’s position and the position of the map container with respect to the datum surface. Many map projections are possible, but out of infinite theoretical possibilities, the important map projections are discussed and few of them are applied. Projections which are important from the point of view of its application can be classified into following categories:

(i) Azimuthal map projection
(ii) Conical map projection
(iii) Cylindrical map projection
(iv) Mercator map projection

Generally map projections are derived for known position of the eye on the main meridian of the datum surface (e.g. the axis of rotation) and target point of contact of the map surface.

Described and derived the mathematical equation of azimuthal map projection which takes into account the viewer’s position at any arbitrary place on the datum surface. The projection for the azimuthal category with arbitrary viewer position is further validated for specific cases of gnomic, stereographic and orthographic projections. The geometric manifestation of the proposed generic azimuthal map projection (GAP) is also described.

2. GEOMETRIC MANIFESTATION OF THE GAP

Let us consider a generalised case of the azimuthal projection where the observer can be anywhere on the surface of the earth as depicted in Fig. 1.

Figure 1, let \(V\) be the generic position of the viewer on the datum surface, which is at \(hR\) distance (where \(R\) is the radius of the earth) from the map surface (SS). The map surface is tangent to the datum surface at point \(F\) (Pole of the earth).

\(FF'\) is the prime meridian where-as \(EE'\) is the equator. The other specifications of the figure are:

- \(O\) is the centre of the earth i.e. scale reduced of the earth
3. MATHEMATICAL FORMULATION OF GAP

Figure 1 shows, the geometrical construction of the generic azimuthal projection is described, where

\[ OF' = hR - R = R(h-1) \]

in \( \Delta OF' \)

\[ \sin \delta = \frac{OF'}{OV} \]

\[ \Rightarrow OV = R(h-1) \csc \delta \]

\[ \Rightarrow VF' = R(h-1) \cot \delta \]

From Pythagoras theorem in \( \Delta VF' \)

\[ VF^2 = VF'^2 + FF'^2 = R^2(h-1)^2 \cot^2 \delta + h^2 R^2 \]

\[ \Rightarrow VF = R \sqrt{(h-1)^2 \cot^2 \delta + h^2} = RL \]

(1)

\[ \text{where } L = \sqrt{(h-1)^2 \cot^2 \delta + h^2} \]

Using sine rule in \( \Delta VFO \)

\[ \frac{VF}{\cos \delta} = \frac{R}{\sin \angle VFO} \]

\[ \Rightarrow \angle VFO = \sin^{-1} \left( \frac{\cos \delta}{VF} \right) \]

\[ \Rightarrow \angle VFO = \sin^{-1} \left( \frac{\cos \delta}{L} \right) \]

\[ \Rightarrow \angle MVF = \frac{\pi}{2} - \delta - \theta \]

\[ \Rightarrow \angle PVF = \alpha + \frac{\pi}{2} - \delta - \theta = \alpha + \eta \]

(4)

(\text{where } \eta = \frac{\pi}{2} - \delta - \theta)

In \( \Delta VPO \)

\[ \angle VPO = \pi - (\pi - \beta - \alpha) = \beta - \alpha \]

\[ \angle VF = \angle OPF + \angle VPO = \frac{\pi}{2} + \frac{\beta}{2} - \alpha \]

(5)

In \( \Delta OPF \), \( D \) is the midpoint of \( PF' \).

So from standard geometry \( OD \) is the perpendicular bisector of \( PF' \) and \( OD \) also bisects \( \angle POF \).

\[ \Rightarrow PD = DF \]

and \( \angle POD = \angle DOF = \frac{\beta}{2} \)

\[ \Rightarrow PD = DF = R \sin \frac{\beta}{2} \]

\[ \Rightarrow PF = 2 R \sin \frac{\beta}{2} \]

(6)

Applying sine rule in \( \Delta PVF \)

\[ \frac{VF}{\sin \angle VPF} = \frac{PF}{\sin \angle PVF} \]

From Eqns (4), (5), (6)

\[ \frac{L}{\cos \left( \frac{\beta}{2} - \alpha \right)} = \frac{2 \sin \frac{\beta}{2}}{\sin (\alpha + \eta)} \]

\[ \Rightarrow L \sin (\alpha + \eta) = 2 \sin \frac{\beta}{2} \left( \cos \frac{\beta}{2} \cos \alpha + \sin \frac{\beta}{2} \sin \alpha \right) \]

(7)

Solving Eqn (7) one can obtain

\[ \tan \alpha = \left[ \frac{\sin \beta - L \sin \eta}{L \cos \eta - 2 \sin^2 \frac{\beta}{2}} \right] \]

(8)

Now radial distance \( \rho = PF = P'N + NF = P'N + VF' \)

\[ \Rightarrow \rho = hR \tan \alpha + R(h-1) \cot \delta \]

From Eqn (8)

\[ \rho = hR \left[ \frac{\sin \beta - L \sin \eta}{L \cos \eta - 2 \sin^2 \frac{\beta}{2}} \right] + R(h-1) \cot \delta \]

(9)
Therefore the transformation from spherical coordinate to the Cartesian coordinate is given by:

\[ X = \rho \sin (\lambda - \lambda_0) \]

\[ Y = \rho \cos (\lambda - \lambda_0) \]

\[ X = hR \left[ \frac{\sin \beta - L \sin \eta}{\cos \eta - 2 \sin^2 \frac{\beta}{2}} \right] + R(h - 1) \cot \delta \sin (\lambda - \lambda_0) \]

\[ Y = hR \left[ \frac{\sin \beta - L \sin \eta}{\cos \eta - 2 \sin^2 \frac{\beta}{2}} \right] + R(h - 1) \cot \delta \cos (\lambda - \lambda_0) \]

4. DERIVATION OF SPECIFIC PROJECTIONS

There are various special cases of map projections, where the observer position is made fixed for the convenience of calculation. The major map projections are:

(a) Gnomic projection
(b) Stereographic projection
(c) Orthographic projection

4.1 Gnomic projection

In case of gnomic projection the observer is positioned at the centre of the earth. Hence in this case \( h = 1 \) and \( \delta = \frac{\pi}{2} \).

So \( L = \sqrt{(h-1)^2 \cot^2 \delta + h^2} = 1 \)

Substituting \( h = 1, \delta = \frac{\pi}{2} \) and \( L = 1 \) in the Eqs (10) and (11) one can obtain:

\[ X_{\text{Gnomic}} = R \tan \beta \sin (\lambda - \lambda_0) \]

\[ Y_{\text{Gnomic}} = R \tan \beta \cos (\lambda - \lambda_0) \]  

4.2 Stereographic Projection

In stereographic projection the observer is placed at the opposite pole of the map surface. Hence in this case \( h = 2, \delta = \frac{\pi}{2} \).

So \( L = \sqrt{(h-1)^2 \cot^2 \delta + h^2} = 2 \)

Substituting \( h = 2, \delta = \frac{\pi}{2} \) and \( L = 1 \) in Eqs (10) and (11) one can obtain:

\[ X_{\text{Stereographic}} = 2R \left[ \frac{\sin \beta - 2 \sin 0}{2 \cos 0 - 2 \sin^2 \frac{\beta}{2}} \right] + \left( h - 1 \right) \cot \delta \sin (\lambda - \lambda_0) \]

\[ Y_{\text{Stereographic}} = 2R \left[ \frac{\sin \beta - 2 \sin 0}{2 \cos 0 - 2 \sin^2 \frac{\beta}{2}} \right] + \left( h - 1 \right) \cot \delta \cos (\lambda - \lambda_0) \]

4.3 Orthographic Projection

In orthographic projection the observer lies on the polar line of the earth, but at infinite distance. Hence in this case \( h = \infty \) and \( \delta = \frac{\pi}{2} \). So we have to apply the limiting case.

Applying limiting conditions in Eqn (2):

\[ \theta = \lim_{h \to \infty} \left( \frac{\cos \delta}{\sqrt{(h-1)^2 \cot^2 \delta + h^2}} \right) = 0 \]

Since \( \delta = \frac{\pi}{2} \)

\[ \Rightarrow \eta = \frac{\pi}{2} - \delta - \theta = 0 \]  

\[ \rho = hR \left[ \frac{\sin \beta - L \sin \eta}{\cos \eta - 2 \sin^2 \frac{\beta}{2}} \right] + R(h - 1) \cot \delta \]

\[ = hR \left[ \frac{\sin \beta - L \sin \eta}{\cos \eta - 2 \sin^2 \frac{\beta}{2}} \right] + R \left[ \frac{\sin \beta}{L - 2 \sin^2 \frac{\beta}{2}} \right] \]

Now

\[ \lim_{h \to \infty} \left( \frac{(h-1)^2 \cot^2 \delta + h^2}{h} \right) = h \sqrt{\cot^2 \delta + 1} \]

\[ = \csc^2 \delta = 1 \quad (\because \delta = \frac{\pi}{2}) \]

\[ \Rightarrow \rho = R \sin \beta \]

\[ X_{\text{Orthographic}} = R \sin \beta \sin (\lambda - \lambda_0) \]

\[ Y_{\text{Orthographic}} = R \sin \beta \cos (\lambda - \lambda_0) \]
5. INVERSE TRANSFORMATION OF GAP
In the previous section the forward transformation of GAP was described. Similarly, the inverse transformation (i.e., the transformation from Cartesian coordinate system to spherical datum) is also of great importance. So we have to find the value of φ and λ as a function of X and Y.

Here, the inverse transformation for the special types of generic azimuthal map projection (GAP) has been derived.

5.1 Inverse Transformation for Gnomonic Projection
From Eqns (12) and (13)
\[ X_{\text{Gnomic}} = R \tan \beta \sin (\lambda - \lambda_0) = X_G \]
\[ Y_{\text{Gnomic}} = R \tan \beta \cos (\lambda - \lambda_0) = Y_G \]
\[ X_G^2 + Y_G^2 = R^2 \tan^2 \beta \]
\[ \frac{X_G^2 + Y_G^2}{R^2} = \tan^2 \beta \]
\[ \Rightarrow \phi = \cot^{-1} \left( \frac{\sqrt{X_G^2 + Y_G^2}}{R} \right) \]

5.3 Inverse Transformation for Stereographic Projection
From Eqns (14) and (15)
\[ X_{\text{Stereographic}} = 2R \tan \beta \sin (\lambda - \lambda_0) = X_S \]
\[ Y_{\text{Stereographic}} = 2R \tan \beta \cos (\lambda - \lambda_0) = Y_S \]
\[ X_S^2 + Y_S^2 = 4R^2 \tan^2 \beta \]
\[ \Rightarrow \alpha = 2 \tan^{-1} \left( \frac{\sqrt{X_G^2 + Y_G^2}}{2R} \right) \]
\[ \Rightarrow \phi = \frac{\alpha}{2} - 2 \tan^{-1} \left( \frac{\sqrt{X_G^2 + Y_G^2}}{2R} \right) \]

5.4 Inverse Transformation for Orthographic Projection
From Eqns (10) and (11)
\[ X_{\text{Orthographic}} = R \sin \beta \sin (\lambda - \lambda_0) = X_O \]
\[ Y_{\text{Orthographic}} = R \sin \beta \cos (\lambda - \lambda_0) = Y_O \]
\[ X_O^2 + Y_O^2 = R^2 \sin^2 \beta \]
\[ \Rightarrow \phi = \cos^{-1} \left( \frac{\sqrt{X_G^2 + Y_G^2}}{R} \right) \]

5.6 CONCLUSIONS
A generic map projection method such as GAP will prove to be quite useful to generate relatively more accurate digital map corresponding to spatial data in comparison to traditional map projection methods. GAP has the following advantages in comparison to the traditional map projection methods (a) that this projection can generate digital maps for users at any part of the globe for accurate location, navigation, and measurement. (b) The map projection is independent of the position of the viewer unlike traditional map projection methods which demand that the viewer’s position be fixed at the centre, perspective of the globe. So we can get the projection of any point of the datum surface with respect to any point of reference.

REFERENCES

CONTRIBUTORS

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