Calculation Method for Three Dimensional Turbulent Boundary Layers

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Abstract. A momentum integral method is developed to predict the growth of three dimensional turbulent boundary layers. Four equations, have been derived in streamline coordinates and solved: (a) momentum integral equation in streamwise direction (b) momentum integral equation in crosswise direction (c) moment of momentum equation in streamwise direction (d) a skin friction equation. The streamwise velocity profile is assumed to be a combination of the logarithmic law of the wall and a linear law of the wake while Mager's expression is used to represent cross flow velocity profile.

1. Basic Equations

Timman\(^1\) has derived equations of motion and continuity in streamline coordinates wherein the longitudinal direction (\(x_1\) direction) follows a streamline outside the boundary layer and is termed streamwise direction. Crosswise direction (\(x_3\) direction) is described by the lines orthogonal to these streamlines and parallel to the surface. Transverse direction (\(x_2\) direction) is defined by the outward normal to the surface, (Fig. 1). In three dimensional boundary layers developing on surfaces of arbitrary shapes velocity profile is skewed. At every \(x_3\) location the velocity vector \(\vec{U}\) changes its direction and can be resolved into streamwise (\(u_1\)) and crosswise (\(u_3\)) components.

The skew angle \(\alpha\) made by \(\vec{U}\) with \(x_1\) direction gradually approaches zero magnitude as the edge of the layer is reached, it attains its maximum value \(\alpha_0\) at the wall. The equations for an incompressible steady mean flow are

Equation of motion in streamwise direction

\[
\frac{u_1}{h_1} \frac{\partial u_1}{\partial x_1} + \frac{u_3}{h_2} \frac{\partial u_1}{\partial x_2} + \frac{u_2}{h_3} \frac{\partial u_1}{\partial x_3} + k_1 u_1 u_2 - k_2 u_1^2 = -\frac{1}{\rho} \frac{\partial p}{\partial x_1} + \frac{1}{\rho} \frac{\partial \tau_{13}}{\partial x_2}
\]  

(1)

Equation of motion in crosswise direction

\[
\frac{u_1}{h_1} \frac{\partial u_2}{\partial x_1} + \frac{u_3}{h_2} \frac{\partial u_2}{\partial x_2} + \frac{u_2}{h_3} \frac{\partial u_2}{\partial x_3} + k_1 u_1 u_2 - k_2 u_2^2 = -\frac{1}{\rho} \frac{\partial p}{\partial x_2} + \frac{1}{\rho} \frac{\partial \tau_{23}}{\partial x_3}
\]  

(2)

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Continuity equation

\[
\frac{1}{h_1 h_3} \left[ \frac{\partial (h_2 u_3)}{\partial x_1} + \frac{\partial (h_1 u_2)}{\partial x_3} + \frac{\partial (h_1 h_3 u_3)}{\partial x_3} \right] = 0
\]  

(3)

In a turbulent boundary layer, the instantaneous velocity components are denoted as \( u_1 + u'_1, \ u_2 + u'_2 \) and \( u_3 + u'_3 \) along \( x_1, \ x_3 \) and \( x_3 \) directions respectively where \( u_1, \ u_2 \) and \( u_3 \) are mean values and \( u'_1, \ u'_2, \ u'_3 \) are the corresponding fluctuations. \( p \) denotes the mean static pressure and \( \rho \) the mean density of the fluid. \( h_1, \ h_2 \) and \( h_3 \) are the three Lame coefficients along \( x_1, \ x_2 \) and \( x_3 \) respectively. \( \tau_{13} \) and \( \tau_{23} \) are shear stress components in \( x_1 \) and \( x_2 \) directions respectively.

\[
\tau_{13} = \mu \frac{\partial \bar{u}_1}{\partial x_3} - \rho \bar{u}'_1 \bar{u}'_3 \quad \text{and} \quad \tau_{23} = \mu \frac{\partial \bar{u}_2}{\partial x_3} - \rho \bar{u}'_2 \bar{u}'_3
\]  

(4)

where \( \mu \) is absolute viscosity. Terms \( -\rho \bar{u}'_1 \bar{u}'_3 \) and \( -\rho \bar{u}'_2 \bar{u}'_3 \) are Reynolds stresses. The geodesic curvatures \( k_1 \) and \( k_2 \) are

\[
k_1 = \frac{1}{h_1 h_2} \frac{\partial h_3}{\partial x_1} \quad \text{and} \quad k_2 = \frac{1}{h_2 h_3} \frac{\partial h_1}{\partial x_3}
\]  

(5)

Assuming the flow in the free stream to be irrotational the following relationship is obtained

\[
\frac{1}{U_\infty} \frac{\partial U_\infty}{h_3 \partial x_3} = -k_2
\]  

(6)

\( U_\infty \) is the local free stream velocity.

2. Derivation of Equations

Integrating Eqn. (1) with respect to \( x_3 \) within the limits of 0 to \( \infty \), after eliminating \( u_3 \) through Eqn. (3), the streamwise momentum integral equation is obtained.
\[
\frac{\partial \theta_{x1}}{\partial s} + \frac{1}{U_{\infty}} \frac{\partial U_{\infty}}{\partial s} [2\theta_{x1} + \delta_{1}^*] + \frac{\partial \theta_{x2}}{\partial n} + \frac{1}{U_{\infty}} \frac{\partial U_{\infty}}{\partial n} [\delta_{2}^* - 2\theta_{x2}] \\
+ k_{1}(\theta_{x1} + \theta_{x2}) + k_{3}(\delta_{2}^* - 2\theta_{x2}) = \frac{\tau_{x1}}{\rho U_{\infty}^2} = \frac{C_{f1}}{2} = \omega \cdot a^* \tag{7}
\]

where

\[
\delta_{1}^* = \int_{0}^{\infty} \left( 1 - \frac{u_{x1}}{U_{\infty}} \right) dx_{2} \tag{8}
\]

\[
\theta_{x1} = \int_{0}^{\infty} \frac{u_{x1}}{U_{\infty}} \left( 1 - \frac{u_{x1}}{U_{\infty}} \right) dx_{2} \tag{9}
\]

\[
\theta_{x2} = \int_{0}^{\infty} \frac{u_{x2}}{U_{\infty}} \left( 1 - \frac{u_{x2}}{U_{\infty}} \right) dx_{2} \tag{10}
\]

\[
\theta_{x3} = \int_{0}^{\infty} \left( \frac{u_{x3}}{U_{\infty}} \right)^2 dx_{2} \tag{11}
\]

\[
\theta_{x4} = \int_{0}^{\infty} \frac{u_{x4}U_{\infty}}{U_{\infty}} dx_{2} \tag{12}
\]

\[
\delta_{2}^* = \int_{0}^{\infty} \frac{u_{x3}}{U_{\infty}} dx_{2} \tag{13}
\]

\(\tau_{x1}\) and \(C_{f1}\) are wall shear stress and skin friction coefficient in \(x_1\) direction respectively while \(\omega = \sqrt{\frac{C_{f1}}{2}}\) is a skin friction parameter. Terms \(\partial \theta(= h_{1}\partial x_{1})\) and \(\partial n(= h_{2}\partial x_{2})\) are infinitesimal arc lengths along \(x_1\) and \(x_2\) directions respectively.

Similarly integrating Eqn. (2) with respect to \(x_2\) between the limits 0 to \(\infty\) and using Eqn. (3) to eliminate \(u_{x3}\), crosswise momentum integral equation is obtained

\[
\frac{\partial \theta_{x3}}{\partial s} + \frac{\partial \theta_{x3}}{\partial n} + \frac{2}{U_{\infty}} \frac{\partial U_{\infty}}{\partial s} \theta_{x3} + 2k_{1} \theta_{x1} + \frac{2}{U_{\infty}} \frac{\partial U_{\infty}}{\partial n} \theta_{x3} \\
+ k_{3}(\theta_{x3} + \delta_{2}^* + \theta_{x2}) = -\frac{\tau_{x3}}{\rho U_{\infty}^2} = \frac{C_{f3}}{2} = -\epsilon_{0} \omega \cdot a^* \tag{14}
\]

where \(\tau_{x3}\) and \(C_{f3}\) are respectively the wall shear stress and skin friction coefficient in \(x_3\) direction. It is assumed that for small values of \(a_{0}\)

\[
\epsilon_{0} = \tan a_{0} \approx \frac{\tau_{x3}}{\tau_{x1}} \tag{15}
\]
Multiplying Eqn. (1) by \( \frac{u_2}{x_2} \) and substituting for \( u_2 \) through Eqn. (3) and integrating the resulting equation with respect to \( x_2 \) between the limits 0 to \( \infty \), moment of momentum equation in streamwise direction is obtained

\[
\frac{\partial}{\partial s} \left[ U_{co}^2 x_2 \left( 1 - \frac{u_1}{U_{co}} \right) x_2 \right] + U_{co} \int_0^x \left[ 1 - \frac{u_2}{U_{co}} \right] \frac{\partial}{\partial s} \left\{ U_{co} x_2 \right\} \left[ \int_0^x u_2 d x_2 \right] + \int_0^x \left[ \int_0^x u_2 d x_2 \right] \right] + \int_0^x \left[ \int_0^x u_2 d x_2 \right] \right] d x_2
\]

Simplification of Integral Equations

Equations (7), (14) and (16) are simplified by guessing the order of magnitude of the various integral quantities. It is assumed that variation of integral quantities in streamwise direction is large as compared with the variation in crosswise direction; also \( u_0 \ll u_1 \).

Velocity Profiles in Three Dimensional Turbulent Boundary Layers

In the present method it is assumed that the variation of \( u_1 \) in \( x_2 \) direction is given by Rotta's velocity profile:

\[
\frac{u_1}{u_0} = \frac{1}{k} \left( \ln \frac{U_0 x_2}{v} \right) + \frac{A}{k} (2\eta) + C \text{ for } 0 \ll x_2 \ll 8
\]

and

\[
\frac{u_1}{U_{co}} = 1 - \frac{\infty}{k} [2A(1 - \eta) - lm]
\]
where \( \delta = x_3/\xi_1 = 0.995 \ U_{\infty} \) is absolute boundary layer thickness, \( \alpha = (\alpha_{\xi_1/\rho})^{1/2} \) is shear velocity, \( k \) is von kármán constant \((0.41)\), \( \nu \) is kinematic viscosity, \( \eta \) is non-dimensionalised distance \((= x_3/\delta)\), \( C \) is a constant and \( \ln \) is Napierian logarithm. \( A \) is a free parameter.

Mager’s\(^a\) profile has been taken to represent the velocity in crosswise direction

\[
\frac{U_{\infty}}{U_{\infty}} = \epsilon_0 (1 - \eta)^A \frac{U_{\infty}}{U_{\infty}}
\]  

(19)

**Final Form of Integral Equations**

Substitution of Rotta’s and Mager’s profiles into the simplified integral equations yield and streamwise momentum integral equation

\[
\begin{align*}
\omega(G_1 - \omega G_3) \frac{\partial \delta}{\partial s} + \delta(G_1 - 2G_3 \omega) \frac{\partial \omega}{\partial s} + \omega \delta \left( \frac{1}{k} - \alpha G_4 \right) \frac{\partial A}{\partial s} \\
+ \frac{2}{U_{\infty}} \frac{\partial U_{\infty}}{\partial s} \delta \omega(1.5G_1 - \omega G_3) + k_3 \delta \omega(G_1 - \omega G_3) - \omega^a = 0
\end{align*}
\]  

(20)

where

\[
G_1 = \frac{1}{k} (A + 1)
\]  

(21)

\[
G_2 = \frac{1}{k} (1.33A^2 + 3A + 1)
\]  

(22)

\[
G_3 = \frac{1}{k} (2.06A + 3)
\]  

(23)

**Crosswise Momentum Integral Equation**

\[
\begin{align*}
\delta \epsilon_0 (2G_3 \omega - G_3) \frac{\partial \omega}{\partial s} + (dn) \left( \epsilon_0 \frac{\partial \delta}{\partial s} + \delta \omega \epsilon_0 \left( \omega G_3 - \frac{1}{k} \right) \frac{\partial A}{\partial s} \\
+ \left[ \delta \epsilon_0 (dn) \frac{2}{U_{\infty}} \frac{\partial U_{\infty}}{\partial s} + 2k_3 \delta \epsilon_0 (dn) + k_3 \delta \omega (2G_1 - \omega G_3) + \epsilon_0 \omega^a \right] \\
+ \delta (dn) \frac{\partial \epsilon_0}{\partial s} = 0
\end{align*}
\]  

(24)

where

\[
G_4 = \frac{1}{k} (A + 1.22)
\]  

(25)

\[
G_6 = \frac{1}{k^3} (0.793A^2 + 2.084A + 1.574)
\]  

(26)

\[
(dn) = (0.33 - \omega G_3 + \omega^a G_6)
\]  

(27)
Streamwise Moment of Momentum Equation

\[ \delta^2 (G_7 - \omega G_9) \frac{\partial \omega}{\partial s} + \delta^2 \omega (G_9 - \omega G_{10}) \frac{\partial A}{\partial s} + \delta \omega (4G_7 - \omega G_9) \frac{\partial B}{\partial s} \]

\[ + \delta \omega \frac{2}{U_{\infty}} \frac{\partial U_{\infty}}{\partial s} (2G_7 - 0.5\omega G_9) + k_1 \delta^2 \omega (2G_7 - \omega G_{11}) = \frac{\delta}{U_{\infty}^2} \int_0^1 \frac{\tau_{12}}{\rho} \, d\eta \]

(28)

where

\[ G_7 = \frac{1}{k} \left( 0.33A + 0.25 \right) \]
(29)

\[ G_9 = \frac{1}{k^2} \left( 1.16A^2 + 2.112A + 1 \right) \]
(30)

\[ G_9 = \frac{0.33}{k} \]
(31)

\[ G_{10} = \frac{1}{k^2} \left( 1.16A + 0.945 \right) \]
(32)

\[ G_{11} = \frac{1}{k^2} \left( A^2 + 1.556A + 0.75 \right) \]
(33)

Clauser's eddy viscosity model along with Rotta's profile have been used to determine the 'Shear Stress Integral' appearing on the right hand side of the Eqn. (28)

\[ \frac{\delta}{U_{\infty}^2} \int_0^1 \frac{\tau_{12}}{\rho} \, d\eta = \omega^2 \left[ \gamma_0 (1 + 2A - 3\eta) - A\eta \gamma_0 \right] \]
(34)

\[ \gamma_0 = \frac{\alpha}{k} \left( 1 + A \right) \]
(35)

\( \gamma_0 \) is the non-dimensionalised distance at which the Clauser's inner layer meets the outer layer and \( \alpha (= 0.018) \) is a constant.

Skin Friction Equation

Equation for skin friction has been derived directly from the streamwise velocity profile, Eqn. (17)

\[ \frac{1}{\delta} \frac{\partial \delta}{\partial s} + 2 \frac{\partial A}{\partial s} \frac{\partial \omega}{\partial s} \left[ \frac{1}{\omega} + \frac{0.41}{A} \right] = - \frac{12 U_{\infty} \tau_{12}}{U_{\infty} \delta s} \]
(36)

Numerical Solution

The equations (20), (24), (28) and (36) form a set of four equations involving \( \delta, A, \omega \) and \( \epsilon_0 \) which have been solved by a Fourth order Runge Kutta method. In order to assess the viability of the calculation method developed, the experimental data of Wakhaloo\(^{a, b}\) has been used. Fig. 2 shows the pattern of
external streamlines obtained by Wakhaloo in his experiment on three dimensional turbulent boundary layers developing in an isolated compressor cascade passage. Results pertaining to one of the streamlines ICP2 are shown in Figs. 3, 4 and 5.

3. Results

Fig. 3 shows the variation of $\delta$, $Cf_i$ and $H \ (= \frac{\delta^*_i}{\theta_1})$ along the streamline ICP2. The experimental values match closely with the predicted values. Fig. 4 shows the variation of $\delta^*_i$, $\theta_1$ and $\theta_2$ while Fig. 5 shows the variation of $\epsilon_\theta$ and $A$ along the
same streamline. The deviations could be due to combined effect of reasons cited below.

The change of curvature of external streamline for ICP2 occurred in the range \( s = 2.0 \) to \( s = 2.4 \). Within this region, the calculated values exhibit a greater deviation from those obtained experimentally. In this zone s shaped cross flow velocity profiles were obtained. Mager’s model of cross flow velocity profile is incapable of representing such profiles, and hence the error.
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The computer program needs the curvatures of external streamlines and equipotentials as input data. Wakhaloo obtained the geometry of external streamlines through potential flow solution assuming that the fluid moved tangentially over the blade surfaces. Actually separation of flow from the blade surfaces was observed which resulted in the alteration of the geometry of the external streamlines, hence the input data to the computer must have been uncertain to an extent.

The magnitude of \( A \) at the starting point of computation was calculated using interpolated values of \( H \) and \( \omega \). This too could introduce some error.

The magnitude of \( \alpha \) in Eqn. (35) was taken to be 0.018, any other suitable value of \( \alpha \) can be chosen to attain a better agreement between predicted and experimental values. Such an approach has already been suggested, Rotta\(^2\).

Terms which have been neglected in the integral equations during simplification are also partly responsible towards the deviations in predicted and experimental data.

4. Conclusion

The calculation method developed in this paper predicts the growth of three dimensional turbulent boundary layers quite satisfactorily.

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