OPTIMUM INTER-TERMINAL TRANSFER TRAJECTORIES

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(Received 11 March '67)

Rocket trajectories in a gravitational field between two terminals with specified velocities at each terminal are investigated with a view to minimize the total velocity increment required in initiating the rocket along the transfer path at the first terminal and in the attainment of the given final velocity at the final terminal. The equations are transformed for transfer between circular orbits and numerical results for Earth—Mars transfer are calculated. Finally particular cases of the above problem are discussed and Stark's results is derived therefrom.

NOMENCLATURE

\[ u_i = \text{given initial velocity} \]

\[ u_1 = \text{departure velocity with which rocket starts along the transfer trajectory just after the impulsive velocity change at the initial terminal.} \]

\[ u_f = \text{given final velocity to be attained.} \]

\[ u_2 = \text{approach velocity of the rocket just before the impulsive velocity change at the final terminal.} \]

\[ \gamma_1 = \text{departure angle (the angle between } u_1 \text{ and the local horizontal).} \]

\[ \gamma_2 = \text{approach angle (the angle between } u_2 \text{ and the local horizontal).} \]

\[ \gamma_i = \text{angle between } u_i \text{ and local horizontal.} \]

\[ \gamma_f = \text{angle between } u_f \text{ and local horizontal.} \]

\[ \rho_1 = \text{radius vector corresponding to initial terminal (Force centre as pole).} \]

\[ \rho_2 = \text{radius vector corresponding to final terminal.} \]

\[ \phi = \text{transfer angle (angle at the force centre between } \rho_1 \text{ and } \rho_2). \]

\[ K = \text{gravitational constant times mass of the attracting body.} \]

Minimization of fuel consumption is an important problem for transfer trajectories. Stark\(^1\) has discussed the problem of finding rocket trajectories between two terminals in space with a view to minimize the velocity increment added to an arbitrary initial velocity at the first terminal, with the restriction that the resulting trajectory passes through the second terminal.

The present paper investigates the optimum trajectory, that is, trajectories of minimum fuel consumption between two terminals in space for a rocket with a given initial terminal velocity so as to attain a specified final terminal velocity. An optimum trajectory here is that which minimizes the total velocity increment at the initial and final terminals. An inverse square gravitational field is taken and the impulsive thrusts are applied at the initial and final terminals only. Particular cases of the above problem are also discussed in the end.
DETERMINATION OF CHARACTERISTIC VELOCITY

The square of the velocity change $\Delta u_i$ at the initial terminal will be

$$ (\Delta u_i)^2 = u_i^2 - 2 u_i u_1 \cos (\gamma_1 - \gamma_i) $$

(1)

If the rocket is required to acquire a velocity $u_f$ at the final terminal, the square of velocity change $\Delta u_f$ is given by

$$ (\Delta u_f)^2 = u_f^2 + u_2^2 - 2 u_f u_2 \cos (\gamma_2 - \gamma_f) $$

(2)

conservation of moment of momentum per unit mass gives

$$ u_1 \rho_1 \cos \gamma_1 = u_2 \rho_2 \cos \gamma_2 $$

(3)

Rocket starts along the transfer path with the departure velocity $u_1$, moves in the gravitational field and acquires the approach velocity $u_2$ at the final terminal. $u_1$ and $u_2$ can be related by the equation

$$ u_2^2 - u_1^2 = 2 K \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) $$

(4)

Substituting from equations (3) and (4), equation (2) becomes

$$ (\Delta u_f)^2 = u_f^2 - 2 u_f \left[ u_1 d \cos \gamma_1 \cos \gamma_f + \sin \gamma_f \left( u_1^2 + 2 K \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \right) \right] + u_2^2 + 2 K \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) = 0 $$

(5)

where

$$ d = \frac{\rho_1}{\rho_2} $$

characteristic velocity $\Delta u$ is then given by

$$ \Delta u = | \Delta u_i | + | \Delta u_f | $$

(6)

where $\Delta u_i$ and $\Delta u_f$ are given by equations (1) and (5) respectively.

ANALYSIS OF OPTIMUM TRAJECTORY

The relation between departure velocity and angle for a trajectory passing through the two specified terminals is expressed by

$$ d = \frac{K}{\rho_1 u_1^2} \left( \frac{1 - \cos \phi}{\cos^2 \gamma_1} \right) + \frac{\cos (\gamma_1 + \phi)}{\cos \gamma_1} \quad / \gamma_1 \leq \frac{\pi}{2} $$

(7)

which can be written as

$$ u_1^2 = \frac{K (1 - \cos \phi \sec^2 \gamma_1)}{\rho_1 (d + \sin \phi \tan \gamma_1 - \cos \phi)} $$

(8)

For optimum transfer trajectory

$$ \frac{\partial (\Delta u)}{\partial \gamma_1} = 0 $$

(9)
Substitution from equations (6), (5) and (1) in equation (9) yields

\[
\frac{1}{\Delta u_i} \left[ P_1 - 2 u_i \left\{ P_2 \cos (\gamma_1 - \gamma_i) - u_1 \sin (\gamma_1 - \gamma_i) \right\} \right] + \frac{1}{\Delta u_f} \left[ P_1 - 2 u_f \left\{ P_2 \cos \gamma_f \left\{ P_2 \cos \gamma_1 - u_1 \sin \gamma_1 \right\} \right\} \right] + \frac{\sin \gamma_f}{2 P_3} \left\{ P_1 (1 - d^2 \cos^2 \gamma_1) + 2 u_i^2 d^2 \cos \gamma_1 \sin \gamma_1 \right\} = 0
\]
(10)

where

\[
P_1 = 2 \left( \frac{K (1 - \cos \phi) \sec^2 \gamma_1}{\rho_1 (d + \sin \phi \tan \gamma_1 - \cos \phi)} \right) \]

\[
P_2 = \frac{K (1 - \cos \phi) \sec^2 \gamma_1}{\rho_1 (d + \sin \phi \tan \gamma_1 - \cos \phi)^2} \left[ 2 \tan \gamma_1 (d - \cos \phi) + \sin \phi (\tan^2 \gamma_1 - 1) \right]
\]

\[
P_3 = \left[ u_i^2 + 2k \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) - u_i^2 d^2 \cos^2 \gamma_1 \right]^{\frac{1}{2}}
\]

(10) can be transformed into an equation of single unknown \( \gamma_1 \) and can be solved numerically for \( \gamma_1 \) which will give optimum departure angle \( \gamma_{10} \) (subscript ‘0’ denotes optimum values) for specified terminals, initial velocity and final velocity to be attained. Having calculated \( \gamma_{10} \), values of \( u_{10} \), \( u_{20} \) and \( \gamma_{20} \) can be obtained from (8), (4) and (3) respectively.

TRANSFER BETWEEN CIRCULAR ORBITS

For transfer between circular orbits obviously \( \gamma_i = \gamma_f = 0 \) and (10) can be easily transformed into

\[
\frac{\cos \gamma_1 (\alpha u_i + \beta u_f)}{A + \sin \phi \tan \gamma_1} + 2 \sin \gamma_1 (\alpha u_i + \beta u_f) = 0
\]
(12)

where

\[
B = \frac{K}{\rho_1} (1 - \cos \phi)
\]

\[
A = d - \cos \phi
\]

\[
C = 2K \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) + u_f^2
\]

\[
\alpha = \left[ \frac{B \sec^2 \gamma_1}{(A + \sin \phi \tan \gamma_1)} + u_i^2 - 2u_i \sqrt{\frac{B}{(A + \sin \phi \tan \gamma_1)}} \right]^{-\frac{1}{2}}
\]

\[
\beta = \left[ \frac{B \sec^2 \gamma_1}{(A + \sin \phi \tan \gamma_1)} + C - 2 \frac{d u_f \sqrt{B}}{(A + \sin \phi \tan \gamma_1)} \right]^{-\frac{1}{2}}
\]

(12) when solved will give the optimum departure angle for the transfer path that requires least fuel consumption in the impulsive change over from the launch orbit to the transfer path at the initial terminal and from the transfer path to the destination orbit.
at the final terminal. From the knowledge of optimum transfer angle other characteristics for the optimum transfer path can be easily calculated with the help of (3), (4) and (8). It is to be observed that in (12), \( \gamma \) signifies the ratio between the radii of launch and destination orbits whereas \( u_i \) and \( u_f \) denote the corresponding orbital velocities.

**NUMERICAL ILLUSTRATION**

**Table 1**

**Numerical Results for Optimum Earth—Mars Transfer**

<table>
<thead>
<tr>
<th>Transfer Angle ( \phi )</th>
<th>Departure Angle ( \gamma_1 )</th>
<th>Approach Angle ( \gamma_2 )</th>
<th>Departure Velocity ( u_1 )</th>
<th>Approach Velocity ( u_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5°</td>
<td>81°45’</td>
<td>54°12’</td>
<td>25.08</td>
<td>4.07</td>
</tr>
<tr>
<td>15°</td>
<td>63°45’</td>
<td>41°48’</td>
<td>26.87</td>
<td>19.46</td>
</tr>
<tr>
<td>45°</td>
<td>30°25’</td>
<td>20°33’</td>
<td>31.10</td>
<td>18.83</td>
</tr>
<tr>
<td>90°</td>
<td>12°45’</td>
<td>10°8’</td>
<td>32.58</td>
<td>21.18</td>
</tr>
<tr>
<td>135°</td>
<td>5°12’</td>
<td>4°42’</td>
<td>32.76</td>
<td>21.48</td>
</tr>
<tr>
<td>180°</td>
<td>0°</td>
<td>0°</td>
<td>32.77</td>
<td>21.4</td>
</tr>
</tbody>
</table>

characteristics for optimum Earth—Mars transfer for various transfer angles are obtained by solving the above equations. A two-dimensional solar system with circular planetary orbits is assumed. A comparison with Stark’s results shows that as expected \( \frac{\Delta u_i}{u_i} \) is always greater than \( \frac{\Delta V_{\text{min}}}{V_1} \).

An observation of Table 2 shows that for any transfer angle, \( \Delta u_i \) is greater than \( \Delta u_f \) and it is encouraging result in the sense of structural economy for space navigation. Since the larger fraction of the fuel will be consumed up at initial terminal leaving the smaller fraction to be carried for use at the final terminal.

**Table 2**

**Terminal Velocity Changes for Optimum Earth—Mars Transfer**

<table>
<thead>
<tr>
<th>Transfer Angle ( \phi )</th>
<th>Velocity Change at Initial Terminal ( \Delta u_i )</th>
<th>Velocity Change at Final Terminal ( \Delta u_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5°</td>
<td>38.63</td>
<td>21.99</td>
</tr>
<tr>
<td>15°</td>
<td>30.01</td>
<td>17.73</td>
</tr>
<tr>
<td>45°</td>
<td>15.92</td>
<td>9.25</td>
</tr>
<tr>
<td>90°</td>
<td>7.50</td>
<td>4.94</td>
</tr>
<tr>
<td>135°</td>
<td>4.47</td>
<td>3.22</td>
</tr>
<tr>
<td>180°</td>
<td>3.02</td>
<td>2.61</td>
</tr>
</tbody>
</table>
The variation of $\frac{\Delta \mu_i}{\mu_i}$ and $\frac{\Delta \mu_f}{\mu_f}$ with respect to transfer angle is shown in Fig. 1.

**PARTicular Cases**

Two special cases of the above problem are (a) rocket trajectories between two given terminals characterized by minimum velocity increment to a given velocity at initial terminal, (b) rocket trajectories between two given terminals characterized by minimum velocity increment to a given velocity at the final terminal.

For trajectories of type (a), characteristic velocity is given by $|\Delta u_i|$ and the departure angle can be obtained by solving the equation

$$P_1 - 2u_i \left\{ P_2 \cos (\gamma_1 - \gamma_i) - u_1 \sin (\gamma_1 - \gamma_i) \right\} = 0$$

(14)

knowing departure angle, departure velocity can be obtained from (8). This is an alternative method to solve the problem discussed in reference. Characteristic velocity for rocket trajectories of type (b) is given by $|\Delta u_f|$ and their departure angle can be obtained by solving the equation

$$P_1 - 2u_f \left[ \frac{\sin \gamma_f}{2 P_3} \left\{ P_1 (1 - d^2 \cos^2 \gamma_1) + 2 u_1^2 d^2 \cos \gamma_1 \sin \gamma_1 \right\} \right] = 0$$

(15)

Having found out departure angle, (8), (4) and (3) will give the departure velocity, approach velocity and approach angle respectively. This is another method of attack for the solution of problem of reference.

Stark’s result can be easily derived from (14) which can be written as

$$2u_1 \frac{\partial u_1}{\partial \gamma_1} - 2u_i \left\{ \frac{u_1}{\partial \gamma_1} \cos (\gamma_1 - \gamma_i) - u_1 \sin (\gamma_1 - \gamma_i) \right\} = 0$$

(16)

Also (8) can be put down as

$$u_1^2 = \frac{B \sec^2 \gamma_1}{A + \sin \phi \tan \gamma_1}$$

(17)

where

$$\frac{\partial u_1}{\partial \gamma_1} = u_1 \tan \gamma_1 - \frac{u_1^3 \sin \phi}{2B}$$

(18)

Substituting (18) in (16) and writing

$$u_1 = X_{opt} \sec \gamma_1$$

$$\tan \gamma_1 = \left[ \frac{B}{X_{opt}^2} \cdot A \right] \frac{1}{\sin \phi}$$

$$u_i \cos \gamma_i = X_1$$

$$u_i \sin \gamma_i = Y_1$$

Fig. 1—Variation of $\frac{\Delta \mu_i}{\mu_i}$ and $\frac{\Delta \mu_f}{\mu_f}$ with respect to transfer angle.
we get Stark's equation

\[(A^2 + \sin^2 \phi) X_{opt}^4 + (A Y_1 \sin \phi - X_1 \sin^2 \phi) X_{opt}^3 + (B Y_1 \sin \phi) X_{opt} - B^2 = 0\]

**ACKNOWLEDGEMENT**

Authors are thankful to Dr. R. R. Aggarwal for useful discussions and to Dr. Kartar Singh for his encouragement in the preparation of this paper.

**REFERENCES**