SHEAR FLOW OF AN ELASTICO-VISCOSOUS FLUID PAST A POROUS FLAT PLATE

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Considering shear flow of an elastico-viscous fluid past a porous flat plate, it has been found that a steady solution for the velocity field is not possible if there is fluid injection at the plate and that the vorticity in the shear flow increases the skin-friction at the plate. If the wall is impermeable, the elastic elements do not affect the velocity field. The thickness of the boundary layer on the plate decreases with the increase of relaxation time, but it increases with the increase of retardation time.

A problem that has recently stimulated much interest concerns the determination of the flow field about a body that is immersed in the stream of a viscous liquid that contains vorticity generated by some external mechanism other than the body. To study this problem in its essential features Li introduced the idealized model of the two-dimensional, unbounded, steady, constant shear flow of an incompressible viscous fluid past an infinitesimally thin, semi-infinite flat plate that is aligned parallel to the oncoming flow. This oncoming flow is essentially the superposition, at constant pressure \( P \), of a uniform flow with constant velocity \( U \) upon a shear flow with a linear velocity distribution. Sakurai extended the problem to the case of an infinite flat plate with uniform suction at the plate. The author solved Sakurai's problem by replacing the viscous liquid by a Maxwellian elastico-viscous liquid. But Maxwell liquid is not a general relation which explains the flow behaviour of a real elastico-viscous liquid. So in this note we have discussed the same problem with the constitutive equation:

\[
P_{ik} + \lambda_1 \left[ \frac{D P_{ik}}{D t} - P_{ij} \delta_{jk} - P_{jk} \delta_{ij} \right] = 2 \mu \left[ \delta_{ik} + \lambda_2 \left( \frac{D \delta_{ik}}{D t} - 2 \delta_{ij} \delta_{jk} \right) \right]
\]  
(1)

where \( \lambda_1 \) and \( \lambda_2 \) are time constants, \( \mu \) is the static or zero shear rate viscosity. The material time derivative \( \frac{D}{D t} \) is

\[
\frac{D a_{ij}}{D t} = \frac{\partial a_{ij}}{\partial t} + v \frac{\partial a_{ij}}{\partial x_k} - w_{ik} a_{kj} + k_j a_{ik}
\]

where

\[
w_{ij} = \frac{1}{2} \left[ \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right]
\]

is the vorticity tensor,

and

\[
d_{ij} = \frac{1}{2} \left[ \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right]
\]

is the rate of strain tensor.

Here \( P_{ik} \) (reduced stress tensor) = \( t_{ik} \) (stress tensor) + \( \rho \delta_{ik} \), where \( \rho \) is a scalar pressure and \( v_i \) is the velocity vector. This is known as Oldroyd B liquid.

X-axis is taken along the plate and Y-axis perpendicular to it. For the flow past an infinite flat plate, conditions will depend on \( y \) only. Hence, the velocity field can be taken as

\[
u = u (y); \quad v = v (y); \quad w = 0
\]

(2)
The boundary conditions of the problem are
\[ y = 0 : u = 0, \quad v = v_o; \quad y \to \infty : u \to U + \gamma y \]  \hspace{1cm} (3)

The velocity field (2) is compatible with the equation of continuity if
\[ \frac{dv}{dy} = 0, \]
which on integration gives
\[ v = \text{constant} = v_o, \]  \hspace{1cm} (4)

\( v_o \) being the constant normal velocity at the plate.

The stress-strain rate relations for the elasto-viscous liquid are reduced to
\[ p_{xx} + \lambda_1 \left[ v_o \frac{dp_{xx}}{dy} - 2 \frac{du}{dy} \right] = -2 \mu \lambda_2 \left( \frac{du}{dy} \right)^2, \]  \hspace{1cm} (5)
\[ p_{xy} + \lambda_1 \left[ v_o \frac{dp_{xy}}{dy} - \frac{du}{dy} p_{yy} \right] = \mu \left[ \frac{du}{dy} + \lambda_2 v_o \frac{d^2u}{dy^2} \right], \]  \hspace{1cm} (6)
\[ p_{yy} + \lambda_1 v_o \frac{dp_{yy}}{dy} = 0 \]  \hspace{1cm} (7)

Equation (7) shows that \( p_{yy} = 0 \) is a particular solution of this.

Hence, putting \( p_{yy} = 0 \) in equation (6) we get
\[ p_{xy} + \lambda_1 v_o \frac{dp_{xy}}{dy} = \mu \left[ \frac{du}{dy} + \lambda_2 v_o \frac{d^2u}{dy^2} \right] \]  \hspace{1cm} (8)

The momentum equations now reduce to
\[ \rho v_o \frac{du}{dy} = - \frac{\partial p}{\partial x} + \frac{\partial p_{xy}}{\partial y}, \]  \hspace{1cm} (9)
\[ 0 = - \frac{\partial p}{\partial y}. \]  \hspace{1cm} (10)

Equation (9) shows that \( \frac{\partial p}{\partial x} \) is a function of \( y \),

Hence, from equation (10)
\[ \frac{\partial p}{\partial y} = 0. \]
\[ \frac{\partial p}{\partial x} = \alpha, \quad \text{a constant} \]  \hspace{1cm} (11)

which gives the pressure distribution
\[ p = \alpha x + \beta \]

where \( \beta \) is a constant.

Eliminating \( p_{xy} \) between (8) and (9), we have
\[ \mu \lambda_2 v_o \frac{d^2u}{dy^2} + (\mu - \lambda_1 \rho v_o^2) \frac{d^2u}{dy^2} - \rho v_o \frac{du}{dy} - \alpha = 0 \]  \hspace{1cm} (12)

The solution of this equation subject to boundary condition \( u = 0 \) at \( y = 0 \) is
\[ u = A (e^{my} - 1) + B (e^{my} - 1) - \frac{\alpha}{\rho v_o} y. \]  \hspace{1cm} (13)
TABLE 1

<table>
<thead>
<tr>
<th>$\frac{K}{R_c}$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.909</td>
<td>..</td>
<td>..</td>
<td>..</td>
</tr>
<tr>
<td>0.4</td>
<td>0.740</td>
<td>0.923</td>
<td>..</td>
<td>..</td>
</tr>
<tr>
<td>0.6</td>
<td>0.588</td>
<td>0.708</td>
<td>0.943</td>
<td>..</td>
</tr>
<tr>
<td>0.8</td>
<td>0.435</td>
<td>0.666</td>
<td>0.813</td>
<td>0.946</td>
</tr>
</tbody>
</table>

where $A$ and $B$ are constants of integration and

$$\begin{align*}
m_1 & = \frac{v_o}{2\nu} \frac{1}{K_c} \left[ -\left( 1 - R_c \right) \pm \left\{ \left( 1 - R_c \right)^2 + 4 K_c \right\}^{\frac{1}{2}} \right] \\

\end{align*}$$

where

$$R_c = \frac{\lambda_1}{\mu} \rho v_o$$

and

$$K_c = \frac{\lambda_2}{\mu} \rho v_o^2$$

Since

$$\left[ \left( 1 - R_c \right)^2 + 4 K_c \right]^{\frac{1}{2}} > \left( 1 - R_c \right),$$

the expression in the square bracket in (14) will have one positive value and the other negative value.

Now we shall study two cases:

(i) $v_o < 0$, which corresponds to fluid suction at the plate; and (ii) $v_o > 0$, which corresponds to fluid injection at the plate.

Case (I) $v_o < 0$

In this case $m_1 < 0$ and $m_2 > 0$. From the condition at infinity we have

$$B = 0, \quad A = -U \quad \text{and} \quad \alpha = \frac{\alpha}{\rho v_o}.$$  

Hence

$$u = U + \alpha y \quad \to \quad \text{U} e^{m_1 y}$$

If $R_c = K_c$, the solution reduces to viscous liquid case.

If $\delta$ is the order of boundary layer thickness, $\frac{\delta v_o}{\nu} \sim 2 \frac{K_c}{\left[ \left( 1 - R_c \right)^2 + 4 K_c \right]} - (1 - R_c)$ and its value for different values of $R_c$ and $K_c$ is given in Table 1, $R_c$ being greater than $K_c$ by definition of the fluid. Table 1 shows that the boundary layer thickness decreases with the increase in relaxation time, but increases with the retardation time.
The skin-friction at the plate $\tau_\circ$ is

$$\tau_\circ = -\left[ \frac{\alpha \mu}{\rho v_\circ} + \rho v_\circ U \right]$$

which shows that the shear stress at the wall is not affected by the elasticity of the fluid but it is not the same as in the ordinary hydrodynamic flow in the absence of vorticity. In shear flow the skin-friction increases.

Case (II) $v_\circ > 0$

In this case, $m_1 > 0$ and $m_2 < 0$. From the condition at infinity, we have

$$A = 0, \quad B = -U \quad \text{and} \quad \eta = -\frac{\alpha}{\rho v_\circ}$$

Hence from (13)

$$u = U + \eta y - U e^{m_2 y}$$

(17)

If $R_c = K_c = 0$, which corresponds to viscous case, we obtain $m_2$ to be infinity which proves that in viscous case a solution is not possible if there is fluid injection at the plate.

The skin-friction at the plate $\tau_\circ$ in this case is also as given in (16). But since there is fluid injection at the plate, $v_\circ > 0$ and $\tau_\circ$ becomes negative which is clearly impossible. Hence, the only possible case for maintaining a laminar motion is to suck out fluid at the plate.

If there is no vorticity in the oncoming flow, we can put $\alpha = 0$ in our equations and the flow over a flat plate in the absence of a pressure gradient comes as particular case.

If $v_\circ = 0$, i.e., no suction is assumed equation (12), subject to the boundary conditions, (3) its has only the plane Poiseuille flow as solution

$$u = \frac{\alpha}{2\mu} y^2 + \beta y, \quad \beta = \text{constant}$$

It is, therefore, interesting to note that the velocity field is affected by the elasticity of the fluid only if the suction is present.

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REFERENCES

2. SAKURAI, T., \textit{J. ibid}, 24 (1957), No. 4, 1