ON PARTICULAR SOLUTIONS OF $\nabla^4 \Phi = 0$ AND $E^4 \Phi = 0$.  

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ABSTRACT  

The correspondence between the particular solutions of the equations $\nabla^4 \Phi = 0$ and $E^4 \Phi = 0$ are pointed out. The solutions obtained already by Bhatnagar are compared. An elementary discussion of the operational equation $[F_1 F_2 (L_1 + L_2)] \Phi = 0$ is presented. The operations $E_v^2$, $E_v^4$, $H_v^2$ and $H_v^4$ are introduced.  

INTRODUCTION  

The Laplacian operator is denoted by $\nabla^2$ and can be identified with $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in the cartesian system (rectangular).  

The operator $E^2$ stands for $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \omega^2} - \frac{1}{\omega} \frac{\partial}{\partial \omega}$. Recently, some particular solutions were obtained for the equation $E^4 \Phi = 0$ by Bhatnagar. The correspondence between the particular solution of $\nabla^4 \Phi = 0$ and $E^4 \Phi = 0$ was not brought forth in the paper, or at least was not pointed out and hence some results relating the particular solutions of the biharmonic equation and $E^4 \Phi = 0$ are presented here.  

Considering the operator, $E^2 f(x, \omega) := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial \omega^2} - \frac{1}{\omega} \frac{\partial f}{\partial \omega}$ can be transformed into a form involving the operator $\nabla^2$ by the simple substitution:  

(1)  

$$f = \omega F,$$

$$\frac{\partial f}{\partial \omega} = \omega \frac{\partial F}{\partial \omega} + F \text{ and }$$

$$\frac{\partial^2 f}{\partial \omega^2} = \omega \frac{\partial^2 F}{\partial \omega^2} + 2 \frac{\partial F}{\partial \omega} \text{ and hence }$$

(2)  

$$E^2 f = \omega \left( \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial \omega^2} + \frac{1}{\omega} \frac{\partial F}{\partial \omega} - \frac{F}{\omega^2} \right)$$

We define $\nabla^2$ in $(x, \omega, \phi)$ coordinate system and express $\nabla^2$ as  

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \omega^2} + \frac{1}{r} \frac{\partial}{\partial \omega} + \frac{1}{\omega^2} \frac{\partial^2}{\partial \phi^2} \text{ and }$$

$$\nabla^2 (F e^{i \phi}) = \left( \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial \omega^2} + \frac{1}{\omega} \frac{\partial F}{\partial \omega} - \frac{F}{\omega^2} \right) e^{i \phi}.$$
where $F$ has been assumed to be independent of $\phi$.

Recalling the operation of $\nabla^2$ in the $(x, \omega, \phi)$ system (cylindrical polar coordinates) on functions independent of $\phi$, we may write,

$$
\nabla^2 (x, \omega) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\omega^2} + \frac{1}{\omega} \frac{\partial}{\partial \omega},
$$

It is easily seen that,

$$
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\omega^2} - \frac{1}{\omega} \frac{\partial}{\partial \omega} \right) \left( \omega F \phi \right) = \omega \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\omega^2} + \frac{1}{\omega} \frac{\partial}{\partial \omega} - \frac{1}{\omega^2} \frac{\partial^2}{\partial \phi^2} \right) \left( F \phi \right)
$$

and hence,

$$
E^2 f = E^2 (\omega F) = \omega \left( \nabla^2 (x, \omega) - 1/\omega^2 \right) F.
$$

To obtain a particular solution of the equation $E^2 f = 0$, one only needs a particular solution of $\nabla^2 \phi = 0$, where $\nabla^2$ is the Laplacian operator in $(x, \omega, \phi)$ system and the particular solution $\Phi$ depending on "$\phi" as \omega F \phi \Phi. If such a solution is found, $f$ can be written as $\omega F$;

[Note: The operator $E^2$ is independent of $\phi$]

$$
E^4 f = E^2 (E^2 f)
$$

writing $f = \omega F \phi \Phi$

$$
E^4 (\omega F \phi \Phi) = \omega \Phi E^2 (E^2 (\omega F))
$$

Also, $\omega \nabla^2 (F \phi \Phi) = E^2 (\omega F \phi \Phi)$

consequently

$$
E^4 (\omega F \phi \Phi) = E^2, E^2 (\omega F \phi \Phi)
$$

$$
= E^2, \omega (\nabla^2 x, \omega - 1/\omega^2) \cdot F \phi \Phi
$$

$$
= \omega (\nabla^2 x, \omega - 1/\omega^2) F \phi \Phi
$$

$$
= \omega \nabla^4 F \phi \Phi \text{ and hence}
$$

$$
\omega \nabla^4 (F \phi \Phi) = E^4 (F \phi \Phi) \ [\nabla^2 \equiv \text{Three dimensional Laplacian operator}]
$$

To obtain a particular solution of $E^4 \phi = 0$, one can search for a particular solution of $\nabla^4 \phi = 0$ of the form $\Phi = F \phi \Phi$ ($F$ independent of $\phi$) and 'convert it' to a solution of $E^4 \phi = 0$ by multiplying $F$ by $\omega$. 
To summarise:

consider the operators

\[ E^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \omega^2} - \frac{1}{\omega} \frac{\partial}{\partial \omega} \right) (x, \omega) \]

and

\[ \nabla^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \omega^2} + \frac{1}{\omega} \frac{\partial}{\partial \omega} + \frac{1}{\omega^2} \frac{\partial^2}{\partial \phi^2} \right) (x, \omega, \phi) \]

Assume a solution of the form \( \Phi = F(x, \omega) e^{i\varphi} \) to the equation \( \nabla^2 \Phi = 0 \). Then, a particular solution of \( E^2 \Phi = 0 \) is \( \omega F \); Also if \( F(x, \omega) e^{i\varphi} = \Phi \) is a solution of \( \nabla^4 \Phi = 0 \); \( \omega F = \psi \) is a solution of \( E^4 \psi = 0 \).

**ILLUSTRATIONS**

Obviously, if \( \Phi \) satisfies the equation \( \nabla^2 \Phi = 0 \) it is a solution of \( \nabla^4 \Phi = 0 \), too.

Assuming the \( \phi \) dependence of \( \Phi = Fe^{i\varphi} \) as that of \( e^{i\varphi} \) we seek the solutions of

\[ \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial \omega^2} + \frac{1}{\omega} \frac{\partial F}{\partial \omega} - \frac{F}{\omega^2} = 0 \]

It is well known that

\[ \cosh \quad \text{mx} \quad \begin{cases} AJ_1(m\omega) + BY_1(m\omega) \end{cases} \]

\[ \sinh \]  

are among these.

Hence,

\[ \phi = \omega \quad \cosh \quad \text{mx} \quad \begin{cases} AJ_1(m\omega) + BY_1(m\omega) \end{cases} \]

\[ \sinh \]

are solutions of \( E^2 \Phi = 0 \) and hence of \( E^4 \varphi = 0 \). Also,

\[ 7(a) \quad \phi = \omega \quad \text{mx} \quad A^1 I_1(m\omega) + B^1 K_1(m\omega) ; \]

\[ \sin \]

are solutions of \( E^2 \Phi = 0 \) and \( E^2 \Phi = 0 \).

If \( \omega \) is such that \( E^2 \Phi = f \) where \( f \) is a solution of \( E^2 f = 0 \), \( \Phi \) will be a solution of \( E^4 \Phi = 0 \); Thus we generate some more particular solutions for \( E^4 \Phi = 0 \).

Since the general solution of

\[ \frac{d^2 Y}{d\omega^2} + \frac{1}{\omega} \frac{dY}{d\omega} + \left( m^2 - \frac{1}{\omega^2} \right) Y = AJ,(m\omega) + BY_1(m\omega) \]

\[ = V(m\omega) \], say,
can be expressed as,

\[ J_1(m\omega) \left[ C_1 - \int_a^\omega V(mx)Y_1(mx) \, dx \right] + \int_\beta^\omega Y_1(m\omega) \left[ C_2 + \int_a^\omega V(mx)J_1(mx) \, dx \right] \]

where \( C_1, \, C_2, \, \alpha \) and \( \beta \) are arbitrary constants, some particular solutions of \( E^4 \Phi = 0 \) take the form as expressed by equation 2.18 in the paper by Bhatnagart.\textsuperscript{1}

Simple manipulations result in the new particular solutions given by equation 2.26 of reference 1. Since

\[
\left( \frac{\partial^2}{\partial \omega^2} + \frac{1}{\omega} \frac{\partial}{\partial \omega} - \frac{1}{\omega^2} \right) (A\omega + B/\omega) = 0
\]

and

\[
\frac{\partial^2}{\partial x^2} \left( \frac{\partial^2}{\partial x^2} \right) (a + bx + cx^2 + dx^3) = 0
\]

some more solutions of \( \nabla^4 \Phi = 0 \) can be seen to be of the form,

\( (a + bx + cx^2 + dx^3) (A\omega + B/\omega) e^{i\theta} \) and hence

\[
(a + bx + cx^2 + dx^3) (A\omega^2 + B) = \Phi
\]

is a solution of \( E^4\varphi = 0 \).

\( (a, \, b, \, c, \, d, \, A \) and \( B \) are arbitrary constants)

[cf: equation 2.30 of reference 1]

Also from the solution of

\[
\left( \frac{\partial^2}{\partial \omega^2} + \frac{1}{\omega} \frac{\partial}{\partial \omega} - \frac{1}{\omega^2} \right) f = A\omega + B/\omega,
\]

it is seen that

\[
[C_1 + (C_2 + C_3 \log \omega) \omega^2 + C_4 \omega^4] (a + bx) \text{ is a solution of } E^4 \Phi = 0.
\]

(equation 2.9 of reference 1).

By similar arguments,

\[
\omega \sin \lambda x \frac{I_1(\lambda \omega)}{K_1(\lambda \omega)}; \quad \omega \cosh \lambda x \frac{J_1(\lambda \omega)}{Y_1(\lambda \omega)}
\]

(11) and

\[
x \omega \sin \lambda x \frac{I_1(\lambda \omega)}{K_1(\lambda \omega)}; \quad x \omega \cosh \lambda x \frac{J_1(\lambda \omega)}{Y_1(\lambda \omega)}
\]

can be proved to be particular solutions (cf: equation 2.39 reference 1)

**Polar coordinates:**

The correspondence, that has been proved to exist, between the solutions of \( E^4\varphi = 0 \) and \( \nabla^4 \Phi = 0 \) is particularly useful in helping us to choose the proper particular solutions in other systems of coordinates.
Considering, for instance, spherical polar coordinates for which $\nabla^2 \phi$ assumes the form

$$\nabla^2 \phi = \frac{1}{r^2} \left\{ \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \phi}{\partial \phi^2} \right\},$$

it may first be pointed out that only those solutions of $\nabla^2 \phi = 0$ or $\nabla^4 \phi = 0$ whose dependence on $\phi$ is as $e^{i\phi}$ are of interest in the discussion of $E^4 \Phi = 0$.

It may be recalled that a typical solution of $\nabla^2 \Phi = 0$ which is of the form $F(r, \theta) e^{i\phi}$ is given by

$$F = (A r^n + Br^{-n-1}) \left[ CP'_n (\mu) + DQ'_n (\mu) \right].$$

The particular solutions of $\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) = Ar^n + Br^{-n-1}$

will be of the form $A' r^{n+2} + B' r^{-(n+1)}$ and hence there exist particular solutions of $E^4 \Phi = 0$ which are of the form,

$$\left( \frac{B}{r^n} + A' r^{n+3} + \frac{B'}{r^{n+2}} + Ar^{n+1} \right) \left\{ a_1 P'_n (\mu) + b_1 Q'_n (\mu) \right\} \left( r \sqrt{1-\mu^2} \right).$$

(cf: equation 3.19, reference 1)

In the degenerate case, $n=0$, we may easily prove that the solutions will be

$$B + A' r^3 + B' r^2 + Ar \left( a_1 + a_2 \mu \right).$$

It can be shown that a solution of $\nabla^4 \Phi = 0$

can be given in the form $r^m f(\mu)$ where $f$ is a solution of

$$\left[ \frac{\partial}{\partial \mu} \left( 1 - \mu^2 \right) \frac{\partial}{\partial \mu} \right] + (m-2) (m-1) - \frac{1}{1-\mu^2} \left\{ \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial}{\partial \mu} \right] \right. $n 

$$+ m (m+1) - \frac{1}{1-\mu^2} \right\} f = 0$$

A typical solution will be given by

$$\left\{ \frac{\partial}{\partial \mu} \left( (1 - \mu^2) \frac{\partial}{\partial \mu} \right) + m (m+1) - \frac{1}{1-\mu^2} \right\} f = 0$$

which can be expressed as $A P'_m (\mu) + B Q'_m (\mu)$

Also, If

$$\left\{ \frac{\partial}{\partial \mu} \left( (1 - \mu^2) \frac{\partial}{\partial \mu} \right) + m (m+1) - \frac{1}{1-\mu^2} \right\} P(\mu) = A_1 P'_{m-2}(\mu) + B_1 Q'_{m-2}(\mu),$$

$$\left\{ r^{m+1} f(\mu) \sqrt{1-\mu^2} \right\} \text{ is a typical solution of } E^4 \phi = 0 \text{ with } f(\mu) \text{ of the form}$$

$$A P'_m (\mu) + B Q'_m (\mu) + A_1 P'_{m-2}(\mu) + B_1 Q'_{m-2}(\mu)$$
If $m=1$, it can be easily shown that the solutions so derived will correspond to those obtained already in equation 3·30* of reference 1.

In the particular case of solving the equation $(L_1 + L_2)^2 \phi = 0$ where the linear differential operators $L_1$ and $L_2$ are independent—in the sense that $L_1$ operates only on $f(\xi_1)$ and $L_2$ on $\psi(\xi_2)$ where $f(\xi_1)$ and $\psi(\xi_2)$ are arbitrary functions of the coordinates $\xi_1$ and $\xi_2$. As an illustration, we may note Laplacian $\nabla^2 (x,\omega)$ in cylindrical coordinates, is of this form

$$\nabla^2 (x,\omega) = L_1 + L_2; \quad L_1 = \frac{\partial^2}{\partial x^2};$$

$$L_2 = \frac{\partial^2}{\partial \omega^2} + \frac{1}{\omega} \frac{\partial}{\partial \omega};$$

To obtain particular solutions of $(L_1 + L_2)^2 \phi = 0$ i.e. $(L_1^2 + 2 L_1 L_2 + L_2^2) \phi = 0$, we assume $\phi$ to be of the form $X(\xi_1) Y(\xi_2)$ and dividing the equation by $XY$, one obtains $\frac{L_1^2(X)}{X} + 2 \frac{L_1(Y)}{X} \cdot \frac{L_2(Y)}{Y} = 0$. This suggests that the solution can be determined by solving for $X$ and $Y$ from

$$L_1 \pm \lambda \quad X = 0 \quad \text{and} \quad (L_2 \mp \lambda) \quad Y = g(\xi_2) \quad \text{where} \quad (L_2 \mp \lambda) \quad g = 0$$
or

$$L_2 \pm \lambda \quad Y = 0 \quad \text{and} \quad (L_1 \mp \lambda) \quad X = f(\xi_1) \quad \text{where} \quad (L_1 \mp \lambda) \quad f = 0$$

where $\lambda$ is a constant.

**Illustrations:**

$$L_1 = \frac{\partial^2}{\partial x^2}$$

$$L_2 = \frac{1}{\omega} \frac{\partial}{\partial r} \left( \omega \frac{\partial}{\partial r} \right) - 1/\omega^2$$

If $(L_1 \pm \lambda^2) \quad f = 0$,

$$f(\xi_1) = a \frac{\cos}{\cosh} \lambda X + b \frac{\sin}{\sinh} \lambda X$$

according as the upper or lower sign is chosen. Hence any solution of $(L_1 \mp \lambda) \quad X = f(\xi_1)$ can be expressed as

$$X = \frac{\cos}{\cosh} \lambda x(a_1 + b_1 x) + \frac{\sin}{\sinh} \lambda x(c_1 + d_1 x).$$

If $\lambda = 0$, it can be easily proved that $X = a_1 + b_1 x + c_1 x^2 + d_1 x^3$.

Similarly, if $(L_2 \pm \lambda) \quad Y = 0$,

$$Y = A_1 \frac{I_1}{J_1} (\lambda \omega) + B_1 \frac{K_1}{Y_1} (\lambda \omega) \quad \text{and} \quad (A_1 \omega + B_1/\omega) \quad (\lambda = 0)$$

*It may be pointed out here that the solutions $r^m, r^m \mu^2$ are not "new" as given in reference 1, equation (3·30) but are implicitly stated through equations 3·19 and 3·8 of reference 1 for $n=1$. Also refer Appendix.*
Hence the particular functions ωXY expressed as

\[ I : \omega \left\{ A_1 \frac{I_1}{J_1}(\lambda \omega) + B_1 \frac{K_1}{Y_1}(\lambda \omega) \right\} \left\{ \cos \lambda x (a_1 + b_1 x) + \sin \lambda x (c_1 + d_1 x) \right\} \lambda \neq 0, \]

where \( A_1, B_1, a_1, b_1, c_1, d_1 \) and \( \lambda \) are arbitrary constants and

\[ II : \omega \left( A_1 \omega + B_1 \omega \right) \left( a_1 + b_1 x + c_1 x^2 + d_1 x^3 \right), \ (\lambda = 0) \]

are solutions of \( E^4 (\omega XY) = 0 \).

Also, the solution of

\[ (L_2 \pm \lambda) Y = \left\{ A_1 \frac{I_1}{J_1}(\lambda \omega) + B_1 \frac{K_1}{Y_1}(\lambda \omega) \right\} \lambda \neq 0 \]

can be proved to be

\[ C_1 \frac{I_1}{J_1}(\lambda \omega) + D_1 \frac{K_1}{Y_1}(\lambda \omega) - \left[ \frac{I_1}{J_1}(\lambda \omega) \int \omega \frac{K_1}{Y_1}(\lambda \omega) \left\{ A_1 \frac{I_1}{J_1}(\lambda \omega) + B_1 \frac{K_1}{Y_1}(\lambda \omega) \right\} d\omega \right. \]

\[ - \frac{K_1}{Y_1}(\lambda \omega) \int \omega \frac{I_1}{J_1}(\lambda \omega) \left\{ A_1 \frac{I_1}{J_1}(\lambda \omega) + B_1 \frac{K_1}{Y_1}(\lambda \omega) \right\} d\omega \]

where \( A_1, B_1, C_1 \) and \( D_1 \) are constants. Hence some solutions of \( E^4(\Phi) = 0 \) are

\[ \Phi = \omega XY = \omega \left\{ a_1 \frac{\cos}{\cosh} \lambda x + b_1 \frac{\sin}{\sinh} \lambda x \right\} \delta \left\{ C_1 \frac{I_1}{J_1}(\lambda \omega) + D_1 \frac{K_1}{Y_1}(\lambda \omega) \right\}. \]

III:

\[ - \frac{I_1}{J_1}(\lambda \omega) \int \omega \frac{K_1}{Y_1}(\lambda \omega) \left\{ A_1 \frac{I_1}{J_1}(\lambda \omega) + B_1 \frac{K_1}{Y_1}(\lambda \omega) \right\} d\omega \]

\[ - \frac{K_1}{Y_1}(\lambda \omega) \int \omega \frac{I_1}{J_1}(\lambda \omega) \left\{ A_1 \frac{I_1}{J_1}(\lambda \omega) + B_1 \frac{K_1}{Y_1}(\lambda \omega) \right\} d\omega \]

For \( \lambda = 0 \), \( \omega XY \) will be of the form,

IV: \( (a_1 + b_1 x) \omega \ A_1 \omega + B_1 \omega + C_1 \omega \log \omega + C_2 \omega^3 \)

The solutions I, II, III and IV have already been shown to be solutions of \( E^4 \phi = 0 \) in this paper, equations 7, 8, 9, 10 and elsewhere (reference 1, equations 2.9, 2.18, 2.26 2.31, 2.39 and 2.44)

In case the operator is of the form \( f_1(\xi_1)f_2(\xi_2)(L_1 + L_2) \) where \( L_1 \) and \( L_2 \) are in terms of \( \xi_1 \) and \( \xi_2 \) coordinates only, let us assume a solution of the form \( \frac{x(\xi_1)}{f_1(\xi_1)} \cdot \frac{y(\xi_2)}{f_2(\xi_2)} \) so that

\[ f_1(\xi_1)f_2(\xi_2) \left[ \frac{y}{f_2} L_1 \left( \frac{x}{f_1} \right) + \frac{x}{f_1} L_2 \left( \frac{y}{f_2} \right) \right] = f_1 y L_1 \left( \frac{x}{f_1} \right) + x f_2 L_2 \left( \frac{y}{f_2} \right) \]

\[ f_1 f_2 (L_1 + L_2) \left\{ L_1 \left( \frac{x}{f_1} \right) + x f_2 L_2 \left( \frac{y}{f_2} \right) \right\} = f_1 f_2 \ y L_1 \left[ f_1 L_1 \left( \frac{x}{f_1} \right) \right] \]

\[ + f_1 L_1 \left( \frac{x}{f_1} \right) L_2 \left( \frac{y}{f_2} \right) + f_2 L_2 \left( \frac{y}{f_2} \right) L_1 \left( \frac{x}{f_1} \right) + x L_2 \left( f_2 L_2 \left( \frac{y}{f_2} \right) \right) \]
and hence, representing the operator \( f_1 f_2 (L_1 + L_2) \) by \( f^{\beta_2 \xi} \)
\[
\frac{\int f^{\beta_2 \xi} \left\{ \frac{xy}{f_1 f_2} \right\}}{xy} = 0 \text{ implies}
\]
\[
(19) \quad \frac{L_1 \left\{ f_1 L_1 \left( \frac{x}{f_1} \right) \right\} + L_1 (x) L_2(y)}{x} + \frac{L_2(y)}{y} \cdot \frac{f_1 L_1 \left( \frac{x}{f_1} \right)}{x} + \frac{L_2[f_2 L_2(y/f_2)]}{y} = 0
\]

For the special case when \( f_2(\xi_2) = 1 \)
\[
(20) \quad \frac{L_1 \left\{ f_1 \left[ L_1 \left( \frac{x}{f_1} \right) \right] \right\}}{x} + \frac{L_2(y)}{y} \left\{ \frac{L_1(x)}{x} + \frac{L_1(x/f_1)}{x/f_1} \right\} + \frac{L_2^2(y)}{y} = 0
\]

It can be seen that the condition \( \frac{L_2(y)}{y} = \lambda \) implies,
\[
(21) \quad \frac{L_1 \left\{ f_1 \left[ L_1 \left( \frac{x}{f_1} \right) \right] \right\}}{x} + \lambda \frac{L_1(x)}{x} + \frac{L_1(x/f_1)}{x/f_1} \right\} + \lambda^2 = 0
\]

from which an expression for \( X \) can be derived. If \( \frac{L_1(x)}{x} = a_1 \) and also \( \frac{L_1(x/f_1)}{x/f_1} = a_2 \)
(i.e. if both \( X \) and \( X/f_1 \) are eigen functions of the operator \( L_1 \)), we find that
\[
a_1 a_2 + \frac{L_2(y)}{y} \left\{ a_1 + a_2 \right\} + \frac{L_2^2(y)}{y} = 0
\]
\[
(22) \quad \text{or} \quad (L_2 + a_1) (L_2 + a_2) y = 0
\]

where \( a_1, a_2 \) are constants. This equation can be solved easily for \( Y \) thereafter, and the complete solution thus obtained.

ILLUSTRATION

**Spherical coordinates:**
\[
L_1 \equiv \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) ; \quad f_1 = 1/r^2
\]
\[
= \theta (\theta + 1) \quad ; \quad \theta = \frac{\partial}{\partial z} \text{ where } z = \log r.
\]

For the operator \( L_1 \), \( r^n \) will be an eigen function with the eigen value \( (n^2 + n) \) so that \( r^{n-2} \)
will be an eigen function with the eigen value \( (n^2 - 3n + 2) \).

i.e. \( \frac{L_1 \left( r^n \right)}{r^n} = n(n + 1) \) and \( \frac{L_1 \left( r^n/r^2 \right)}{r^{n-2}} = n^2 - 3n + 2; \)

Correspondingly, one can solve (for \( Y \)) the equation
\[
[L_2 + n(n + 1)] [L_2 + (n - 2)(n - 1)] y = 0
\]

Some of the solutions can be,
\[
Y = A_1 P_n'(\mu) + B_1 Q_n'(\mu) \quad \text{and} \quad A_1 P_{n-2}'(\mu) + B_1 Q_{n-2}'.
\]
In particular, for \( n = 2 \)

\[
Y = A_1 P_2' (\mu) + B_1 Q_2' (\mu)
\]

and

\[
(A_1' + B_1' \mu) \sqrt{1 - \mu^2} \text{ or } A_1 \mu \sqrt{1 - \mu^2} \text{ and } (A_1' + B_1' \mu) / \sqrt{1 - \mu^2},
\]

confining to the Legendre function of the first kind. A solution can therefore be written as

\[
A_1 \mu \sqrt{1 - \mu^2} \text{ or } (A_1' + B_1' \mu) / \sqrt{1 - \mu^2}.
\]

Consequently, the solutions for \( E^4 \phi = 0 \) can, hence be expressed as \( r A_1 \mu (1 - \mu^2) \) and \( r (A_1' + B_1' \mu) \).

\( A_1, A_1' \text{ and } B_1' \text{ : arbitrary constants).} \)

Similarly one can derive the solutions corresponding to other powers.

Discussions of a similar nature can be extended to some other operators as well.

Defining \( E_v^2 = \frac{\partial^2}{\partial \omega^2} - \frac{\nu}{\omega} \frac{\partial}{\partial \omega} + \frac{\partial^2}{\partial x^2} \), it can be proved easily that

\[
(23) \ E_v^2 \left( \frac{\nu + 1}{\omega^2} \right) = \omega \frac{\nu + 1}{2} \left\{ \frac{\partial^2}{\partial \omega^2} + \frac{1}{\omega} \frac{\partial}{\partial \omega} - \frac{(\nu + 1)^2}{4 \omega^2} + \frac{\partial^2}{\partial x^2} \right\}
\]

which corresponds to the operation of the three dimensional Laplacian \( \nabla^2 \) on a function which varies with \( \phi \) as \( e^{i \left( \frac{\nu + 1}{2} \phi \right)} \)

For \( \nu = 1 \), \( E_v^2 \gg E^2 \) discussed in the paper, and \( \frac{\nu + 1}{2} = 1 \); Hence,

\[
(24) \ E_v^2 \left( \frac{\nu + 1}{\omega^2} \ F(\omega, x) e^{i \left( \frac{\nu + 1}{2} \phi \right)} \right) = \omega \frac{\nu + 1}{2} \left( \nabla^2 \left( F e^{i \left( \frac{\nu + 1}{2} \phi \right)} \right) \right)
\]

\[
(24a) \ E^4 \left( \frac{\nu + 1}{\omega^2} \ F(\omega, x) e^{i \left( \frac{\nu + 1}{2} \phi \right)} \right) = \omega \left( \nabla^4 \left( F e^{i \left( \frac{\nu + 1}{2} \phi \right)} \right) \right)
\]

and hence corresponding to a solution of \( \nabla^4 \left( F e^{i \left( \frac{\nu + 1}{2} \phi \right)} \right) = 0 \) with \( F \) independent

\[
\frac{\nu + 1}{2}
\]

\( f \ \phi \), a solution of \( E_v^4 f = 0 \) exists such that \( f = \omega F \). An operator of the form

\[
H_v = \left( \frac{\partial^2}{\partial \omega^2} - \frac{\nu}{\omega} \frac{\partial}{\partial \omega} + \frac{\partial^2}{\partial x^2} - \kappa^2 \right)
\]

will correspond to the Hamiltonian

\[
(25) \ (\nabla^2 - \kappa^2) \text{ since } H_v \left( \frac{\nu + 1}{\omega^2} \ F \right) = \omega \frac{\nu + 1}{2} \left( \nabla^2 - \kappa^2 \right) \left( F e^{i \left( \frac{\nu + 1}{2} \phi \right)} \right)
\]

with \( F \) independent of \( \phi \).

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REFERENCE

APPENDIX

To solve
\[ A(1) \left( 1 - \mu^2 \right) \frac{d^2}{d\mu^2} \left\{ \left( 1 - \mu^2 \right) \mu^2 \right\} + 2a \left( 1 - \mu^2 \right) \mu^2 + \left( a - 3 \right) \left( a - 1 \right) \mu = 0 \]

(cf: equation 3·22, Reference 1)

Denote by \( \theta_1 \) and \( \theta_2 \) the operators \( \left( 1 - \mu^2 \right) \frac{d^2}{d\mu^2} \) and
\[ \left\{ \left( 1 - \mu^2 \right) \frac{d^2}{d\mu^2} - 2\mu \frac{d}{d\mu} - \frac{1}{1 - \mu^2} \right\} \]
respectively; By simple calculations, it may be shown that
\[ \theta_1 \left\{ N(\mu) \sqrt{1 - \mu^2} \right\} = \sqrt{1 - \mu^2}. \theta_2 \left\{ N(\mu) \right\} \]
so that
\[ \theta_1^2 N(\mu) \sqrt{1 - \mu^2} = \theta_1 \left[ \sqrt{1 - \mu^2}. \theta_2^2 N(\mu) \right] = \sqrt{1 - \mu^2}. \theta_2^2 N(\mu) \]

Equation A(1) can be transformed into the form
\[ A(2) \quad \theta_1^2 \mu + 2a \theta_1 \mu + \left( a - 3 \right) (a - 1) \mu = 0. \]

\( N \sqrt{1 - \mu^2} \) is a solution of A(2), provided \( N \) satisfies the equation,
\[ A(3) \quad \left[ \theta_2^2 + 2a \theta_2 + \left( a - 3 \right) (a - 1) \right] N = 0. \]

Denoting the roots \( ( - a \pm \sqrt{4a - 3} ) \) of the equation \( x^2 + 2ax + (a - 3)(a - 1) = 0 \)
by \( a_1 \) and \( a_2 \) respectively, we find that
\[
N = A_1 P'_{n-1} (\mu) + B_1 Q'_{n-1} (\mu) \\
+ A_2 P'_{n-3} (\mu) + B_2 Q'_{n-3} (\mu)
\]
where \( 3 + n(n - 3) \) has been written for \( a \).

The solution of \( (D^2 - 3D + 3) R = aR \)

(cf: equation 3·20, Reference 1)
can be expressed as \( R = C_1 r^n + C_2 r^{-(n - 3)} \).

(note: \( a = 3 + n(n - 3) \)). Hence, a solution of \( E^4 \Phi = 0 \) will be,
\[
\left[ C_1 r^n + C_2 r^{-(n - 3)} \right] [A_1 P'_{n-1} (\mu) + B_1 Q'_{n-1} (\mu) + A_2 P'_{n-3} (\mu) + B_2 Q'_{n-3} (\mu)] \\
\left( \sqrt{1 - \mu^2} \right)
\]

\((A_1, B_1, C_1, A_2, B_2, \text{ and } C_2 \text{ are arbitrary constants})\).

This result has been implicitly stated through equation (17) of this paper.

In particular, for \( n = 1, a = 1 \) and the solutions given by equation 3·30 of reference (1) can be deduced.