An attempt has been made to derive a Fourier series expansion for the $H$-function of two variables recently defined by Verma. This series is analogous to that of other special functions such as the MacRobert's $E$-function, Meijer's $G$-function and Fox's $H$-function of single variable as given by MacRobert, Kesarwani, Parihar, Parashar, Kapoor & Gupta. In the end an integral has been evaluated by making use of this result.

MacRobert, Kesarwani, Parihar and Parashar have proved Fourier series for Fox's $H$-function of single variable. However the Fourier series expansion for $H$-function of two variables has not been derived so far.

The following Fourier series expansion is proposed to be established:

$$
\sum_{r=0}^{\infty} \frac{(k+r)!}{k! r!} H_{n,\nu_1+1,\nu_2, m_1+2, m_2} \begin{pmatrix}
\{a_p, e_p\} & \{r, \bar{r}\}, \{\gamma_q, c_q\}, (0, \bar{h}), (-k-r-1, h); (y', c') \\
\{\delta_q, d_q\} & \left(1 + \frac{k}{2}, h\right), \left(\frac{3}{2} + \frac{k}{2}, h\right); (\beta_q, b_q) \end{pmatrix} \sin (k+2r+1) \theta
$$

where

$$
\sum_{u=0}^{\infty} \frac{\sin \theta (\cos \theta - 1)}{u! (k-u)!} H_{n, \nu, m_1 + 2, m_2} \begin{pmatrix}
\{a_p, e_p\} & \{0, h\}, (-\frac{k}{2} - u, h); (y', c') \\
\{\delta_q, d_q\} & \left(1 + \frac{k+u}{2}, h\right), \left(\frac{3}{2} + \frac{k+u}{2}, h\right); (\beta_q, b_q) \end{pmatrix}
$$

where $0 < \theta < \pi$, $T = \sum_{j=1}^{n} e_j + \sum_{j=1}^{\nu_2} c_j + \sum_{j=1}^{m_1} b_j - \sum_{j=1}^{n+1} e_j - \sum_{j=1}^{\nu_1+1} d_j - \sum_{j=1}^{\nu_2} c_j - \sum_{j=1}^{m_1+1} b_j > 0$, $|\arg x| < \frac{1}{2} T \pi$, and

$$
T' = \sum_{j=1}^{n} e_j + \sum_{j=1}^{\nu_2} c_j + \sum_{j=1}^{m_3} b_j - \sum_{j=1}^{n+1} e_j - \sum_{j=1}^{\nu_1+1} d_j - \sum_{j=1}^{\nu_2} c_j - \sum_{j=1}^{m_1+1} b_j > 0$, $|\arg y| < \frac{1}{2} T' \pi$.

Fox's $H$-function of two variables recently introduced by Verma which is an extension of $G$-function of two variables defined by Agarwal. This $H$-function of two variables does not only includes Fox's $H$-function and the Meijer's $G$-function of single variables as particular cases but also most of special functions of two variables, e.g., Appell's functions, the Whittaker function of two variables etc.

Thus Fox's $H$-function of two variables due to Verma will be defined as follows:

$$
\frac{\Gamma^{\nu_1, \nu_2, m_1, m_2}}{\Gamma^{\nu_1, \nu_2} \Gamma (1 - \nu_1 - \nu_2)} \begin{pmatrix}
\{a_p, e_p\} & \{\gamma_q, c_q\}, (0, \bar{h}), (-k-r-1, h); (y', c') \\
\{\delta_q, d_q\} & \left(1 + \frac{k}{2}, h\right), \left(\frac{3}{2} + \frac{k}{2}, h\right); (\beta_q, b_q) \end{pmatrix} \frac{1}{(2\pi i)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi (\xi + \eta) \psi (\xi, \eta) \alpha^* y^\eta d\xi d\eta, \quad (2)
$$

where

$$
\phi (\xi + \eta) = \prod_{j=1}^{n} \Gamma (1 - a_j + e_j \xi + e_j \eta) \prod_{j=1}^{m} \Gamma (a_j - e_j \xi - e_j \eta) \prod_{j=1}^{s} \Gamma (\delta_j + d_j \xi + d_j \eta) \prod_{j=1}^{t} \Gamma (\beta_j + b_j \xi + b_j \eta)
$$
\[
\psi (\xi, \eta) = \sum_{j=1}^{m_1} \prod_{j=1}^{V_1} \Gamma (\beta_j - b_j \xi) \prod_{j=1}^{V_2} \Gamma (\gamma_j + c_j \xi) \prod_{j=1}^{m_2} \Gamma (\beta_j + b_j \eta) \prod_{j=1}^{V_2} \Gamma (\gamma_j + c_j \eta) \prod_{j=m_1+1}^{V_1} \Gamma (1-\beta_j + b_j \xi) \prod_{j=m_1+1}^{V_2} \Gamma (1-\gamma_j - c_j \xi) \prod_{j=V_1+1}^{V_2} \Gamma (1-\beta_j - b_j \eta) \prod_{j=V_2+1}^{m_2} \Gamma (1-\gamma_j - c_j \eta),
\]

and
\[
0 \leq m_1 \leq q, 0 \leq m_2 \leq q, 0 \leq v_1 \leq t, 0 \leq v_2 \leq t', 0 \leq n \leq p.
\]

The sequence of parameters \((\beta_{m_1}, b_{m_1}, \beta_{m_2}, b_{m_2}, \gamma_{v_1}, c_{v_1}, \gamma_{v_2}, c_{v_2})\) and \((a_n, c_n)\) are such that none of the poles of integrand coincide. The paths of integration are indented, if necessary, in such a manner that all the poles of \(\Gamma (\beta_j - b_j \xi), j=1,2,\ldots,m_1\) and \(\Gamma (\beta_j + b_j \eta), k=1,2,\ldots,m_2\) lie to the right and those of \(\Gamma (\gamma_j + c_j \xi), j=1,2,\ldots,v_1, \Gamma (\gamma_j + c_j \eta), k=1,2,\ldots,v_2, \) and \(\Gamma (1-\alpha_j + c_j \xi + c_j \eta), j=1,2,\ldots,n, \) lie to the left of imaginary axis.

The integral (2) converges if
\[
T = \sum_{j=1}^{n} c_j + \sum_{j=1}^{m_1} b_j \sum_{j=1}^{n+1} c_j - \sum_{j=1}^{s} d_j - \sum_{j=1}^{v_1+1} c_j - \sum_{j=1}^{m_1+1} b_j > 0, |\arg x| < \frac{1}{2} T \pi,
\]
and
\[
T' = \sum_{j=1}^{n} c_j + \sum_{j=1}^{m_2} b_j - \sum_{j=1}^{s} d_j - \sum_{j=1}^{v_2+1} c_j - \sum_{j=1}^{m_2+1} b_j > 0, |\arg y| < \frac{1}{2} T \pi.
\]

We shall give below some results left and use them later on. Askey\(^6\) give with \(\lambda = 1-s\)
\[
(\sin \theta)^{1-s} P_n (\cos \theta) = \sum_{r=0}^{\infty} \frac{2^{2s} (n+r)! \Gamma (n+2-2s) \Gamma (r+s)}{\Gamma (1-s) \Gamma (s) r! n! \Gamma (n+r+2-s)} \sin (n+2r+1) \theta,
\]
where \(s < 1\) and \(0 \leq \theta \leq \pi\) and \(P_n^{(\lambda)} (\cos \theta)\) is given by
\[
(1 - 2r \cos \theta + r^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)} (\cos \theta), r^n,
\]
also Rainville\(^7\)
\[
P_n^{(\lambda)} (z) = \sum_{m=0}^{n} \frac{(2\lambda)_m (z-1)}{m! (n-m)! (\lambda+\frac{1}{2})_m}.
\]
The Legendre duplication formula
\[
\sqrt{(\pi)} \Gamma (2z) = 2^{2z-1} \frac{2z}{\pi} \Gamma (z) \Gamma (z + \frac{1}{2})
\]
Verma\(^8\) gives
\[
H_{p(t:t'), s(q:q')}^{n, v_1, v_2, m_1, m_2} \left[ \begin{array}{c} (a_p, \theta_p) \\ (\gamma_{v_1}, c_{v_1}) \\ (\gamma_{v_2}, c_{v_2}) \\ (\beta_1, d_1) \\ (\beta_2, b_2) \\ (\beta_3, b_3) \\ (\beta_4, b_4) \\ (\beta_5, b_5) \end{array} \right] = -r H_{p(t:t'), s(q:q')}^{n, v_1, v_2, m_1, m_2} \left[ \begin{array}{c} (a_p + c_p \gamma, \theta_p) \\ (\gamma_{v_1} - c_{v_1} \gamma, c_{v_1}) \\ (\gamma_{v_2} - c_{v_2} \gamma, c_{v_2}) \\ (\delta_1, d_1) \\ (\delta_2, d_2) \\ (\delta_3, d_3) \\ (\delta_4, d_4) \\ (\delta_5, d_5) \end{array} \right],
\]
and
\[
H_{p(t:t'), s(q:q')}^{n, v_1, v_2, m_1, m_2} \left[ \begin{array}{c} (a_p, 1) \\ (\gamma_{v_1}, 1) \\ (\gamma_{v_2}, 1) \\ (\beta_1, 1) \\ (\beta_2, 1) \\ (\beta_3, 1) \\ (\beta_4, 1) \\ (\beta_5, 1) \end{array} \right] = G_{p(t:t'), s(q:q')}^{n, v_1, v_2, m_1, m_2} \left[ \begin{array}{c} (a_p) \\ (\gamma_{v_1}) \\ (\gamma_{v_2}) \\ (\beta_1) \\ (\beta_2) \\ (\beta_3) \\ (\beta_4) \end{array} \right],
\]
and
\[
H_{p(t:t'), s(q:q')}^{n, v_1, v_2, m_1, m_2} \left[ \begin{array}{c} (a_p, 1) \\ (\gamma_{v_1}, 1) \\ (\gamma_{v_2}, 1) \\ (\beta_1, 1) \\ (\beta_2, 1) \\ (\beta_3, 1) \\ (\beta_4, 1) \\ (\beta_5, 1) \end{array} \right] = G_{p(t:t'), s(q:q')}^{n, v_1, v_2, m_1, m_2} \left[ \begin{array}{c} (a_p) \\ (\gamma_{v_1}) \\ (\gamma_{v_2}) \\ (\beta_1) \\ (\beta_2) \\ (\beta_3) \\ (\beta_4) \end{array} \right].
\]
Proof of equation (I): On expressing the $H$-function of two variables as Mellin-Barnes type of double integral in L.H.S. of (1) and changing the order of summation and integration as permissible by absolute convergence for stated conditions in (1), the series becomes

$$
\sum_{r=0}^{\infty} \frac{(k+r)!}{k!} \frac{\Gamma \left(1+\frac{k}{2} - h\xi\right) \Gamma \left(\frac{3}{2} + \frac{k}{2} - h\xi\right)}{\Gamma \left(1-h\xi\right) \Gamma \left(1-h\xi\right)} \frac{\Gamma \left(\frac{3}{2} + u - h\xi\right)}{\Gamma \left(1-h\xi\right) \Gamma \left(1-h\xi\right)}
\sin \left(k + 2r + 1\right) \theta \int x^z \sum_{r=0}^{\infty} \frac{(k+r)!}{k!} \frac{\Gamma \left(1+\frac{k}{2} - h\xi\right) \Gamma \left(\frac{3}{2} + \frac{k}{2} - h\xi\right)}{\Gamma \left(1-h\xi\right) \Gamma \left(1-h\xi\right)} \frac{\Gamma \left(\frac{3}{2} + u - h\xi\right)}{\Gamma \left(1-h\xi\right) \Gamma \left(1-h\xi\right)}
\sin \left(k + 2r + 1\right) \theta \int x^z y^n \, d\xi \, d\eta.
$$

Using the result (3) and (6), we have

$$
\int \phi(\xi + \eta) \psi(\xi, \eta) \left[ \frac{\sqrt{\left(\pi\right)}}{\xi+1} (\sin \theta)^{(1-\delta)} \right] P_k (\cos \theta) \int x^z y^n \, d\xi \, d\eta.
$$

Now substituting the value of $P_k (\cos \theta)$ from (5) and then using the result (6), we get

$$
\sum_{u=0}^{k} \frac{\sqrt{\left(\pi\right)} \sin \theta (\cos \theta - 1)^u}{u! (k-u)!} \int \phi(\xi + \eta) \psi(\xi, \eta) \left[ \frac{\sqrt{\left(\pi\right)}}{\xi+1} (\sin \theta)^{(1-\delta)} \right] P_k (\cos \theta) \int x^z y^n \, d\xi \, d\eta
$$

by definition of $H$-function of two variables (2), we get R.H.S. of (1), which completes the proof.

We shall derive here other Fourier series by applying the property of the $H$-function or by specializing the parameters.

(i) The following Fourier series for $H$-function is arrived by using the result (7):

$$
\sum_{r=0}^{\infty} \frac{(k+r)!}{k!} \frac{\Gamma \left(\frac{3}{2} + u - h\xi\right)}{\Gamma \left(1-h\xi\right) \Gamma \left(1-h\xi\right)} H_p, (t + 3 ; t'), s, (q + 3 ; q')
$$

where

$$
x = \sum_{r=0}^{k} \frac{\sqrt{\left(\pi\right)} \sin \theta (\cos \theta - 1)^u}{u! (k-u)!} \int \phi(\xi + \eta) \psi(\xi, \eta) \left[ \frac{\sqrt{\left(\pi\right)}}{\xi+1} (\sin \theta)^{(1-\delta)} \right] P_k (\cos \theta) \int x^z y^n \, d\xi \, d\eta
$$

and

$$
y = \left(1 + \frac{k+u}{2}, h\right), \left(\frac{3}{2} + \frac{k+u}{2}, h\right), (\beta', b'; \beta''', b''').
$$

(9)
On putting $h = 1 = (e_p) = (c_q) = (\sigma r) = (d_s) = (b_y) = (b'_q)$ in (1), and using (8), we get the following result for Meijer $G$-function of two variables:

$$
\sum_{r=0}^{\infty} \frac{(k+r)!}{k!} G_p, (t+3: t'), \left( \delta, (q+3: q') \right)
\begin{align*}
\left( \begin{array}{c}
\frac{a_p}{r} \\
\frac{b_y}{y} \\
\frac{c_q}{r} \\
\frac{d_s}{y} \\
\frac{e_p}{t}
\end{array} \right)
&= \sin \left( k + 2r + 1 \right) \frac{\theta}{\sin^2 \theta}
\left( \begin{array}{c}
\left( \frac{a_p}{r} \\
\frac{b_y}{y} \\
\frac{c_q}{r} \\
\frac{d_s}{y} \\
\frac{e_p}{t}
\end{array} \right)
&= \sin \left( k + 2r + 1 \right) \frac{\theta}{\sin^2 \theta}
\left( \begin{array}{c}
\left( \frac{a_p}{r} \\
\frac{b_y}{y} \\
\frac{c_q}{r} \\
\frac{d_s}{y} \\
\frac{e_p}{t}
\end{array} \right)
\end{align*}
$$

From (1), we can easily deduce the following integral:

$$
\int \frac{\sin \theta \sin \left( k + 2r + 1 \right) \frac{\theta}{\sin^2 \theta} \left( \cos \theta - 1 \right)^n}{u! \left( k - u \right)!} H \sum_{u=0}^{\infty} \frac{\sqrt{\pi} \left( k + r + 1 \right)!}{k! r!} H n, v_1, v_2, m_1 + 2, m_2 \left( \begin{array}{c}
\left( \frac{a_p}{r} \\
\frac{b_y}{y} \\
\frac{c_q}{r} \\
\frac{d_s}{y} \\
\frac{e_p}{t}
\end{array} \right)
&= \sin \left( k + 2r + 1 \right) \frac{\theta}{\sin^2 \theta}
\left( \begin{array}{c}
\left( \frac{a_p}{r} \\
\frac{b_y}{y} \\
\frac{c_q}{r} \\
\frac{d_s}{y} \\
\frac{e_p}{t}
\end{array} \right)
\end{align*}
$$

where $0 \leq \theta \leq \pi$, $T = \sum |x| < 1/2$, $T' = \sum |y| < 1/2$.

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