FLIGHT-PATH CHARACTERISTICS FOR FEW RE-ENTRY TRAJECTORIES

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Approximate analytical solutions on velocity, range, deceleration, maximum deceleration, distance and time of flight, etc. are obtained for object entry in a planetary atmosphere when (i) rate of change of velocity is proportional to the nth power of the velocity and (ii) rate of change of altitude is proportional to the pth power of the velocity. Loh's results for constant deceleration flight and constant rate of change of descent flight are obtained as particular cases. Finally, trajectory characteristics are obtained for constant rate of change of range flight.

NOTATIONS

\[ A = \text{reference area for lift and drag expressions (sq. ft.)} \]

\[ C_D = \text{drag coefficient} \]

\[ D = \text{drag (lb.)} \]

\[ g = \text{acceleration due to gravity (ft./sec}^2) \]

\[ L = \text{lift (lb.)} \]

\[ m = \text{mass (slugs)} \]

\[ R = \text{range (ft.)} \]

\[ R_0 = \text{radius of earth (ft.)} \]

\[ s = \text{distance along flight-path (ft.)} \]

\[ V = \text{velocity (ft./sec.)} \]

\[ \theta = \text{angle of entry} \]

\[ \beta = \text{atmospheric density coefficient such that } \rho = \rho_0 e^{-\beta y} \text{ where } \rho_0 \text{ is atmospheric density at earth's surface and } y = \text{altitude}. \]

The problem of vehicle entry into planetary atmosphere is gradually receiving increasing attention. Gazley, Allen, Eggers and Chappell have considered the case of ballistic-type entry without lift at sufficiently large angles of inclination, where both the gravity force and centrifugal force have been neglected.

Loh has given analytical solutions on the dynamical aspect of various kinds of re-entry trajectories. He has considered the case for constant and for variable lift-drag ratios. His approach is more general and the results of earlier investigators come out as particular cases. The present paper is essentially aimed at generalizing Loh's results for constant deceleration flight and constant rate of change of descent flight and also dealing with a new type of re-entry trajectory. The results for all the above three cases are for variable L/D ratios.

Rate of Change of Velocity Varies as the nth power of Velocity

In a non-rotating two dimensional inertial coordinate system with its origin at the centre of the earth or planet the equations of motion are

\[
\frac{dV^2}{ds} + \frac{2g}{L/D} \left[ \left(1 - \frac{V^2}{gR_0}\right) \cos \theta - \frac{V^2}{g} \left(\frac{d\theta}{ds}\right) \right] = 0
\]

(1)

\[
- \left( \frac{C_D A}{m} \right) \left( \frac{1}{2} \rho V \right) = \frac{dV}{dt} = \frac{1}{2} \frac{dV^2}{ds}
\]

(2)

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For high speed re-entry the gravity component in the drag direction is usually small in comparison with drag itself; consequently, it has been neglected in (2).

The trajectory characteristics at
\[ \frac{dV}{dt} = K_2 V^n \] (3)
are now worked out:

**Density:** From (2) and (3) the expression for density is
\[ \rho = \frac{2m K_2}{C_D A V^{2-n}} \] (4)

**Altitude:** From the altitude-density relation and equation (4)
\[ y = -\frac{1}{\beta} \ln \left( -\frac{2m K_2}{\rho_0 C_D A V^{2-n}} \right) \] (5)

**Time of flight:** Integrating (3),
\[ t = -\frac{1}{K_2} \frac{1}{(1-n)} \left[ V_i^{1-n} - V^1 - n \right] \] (6)
where subscript \( i \), here or elsewhere, denotes initial values.

**Distance along flight-path:** The distance along flight-path using (6) is
\[ s = -\frac{1}{(2-n) K_2} \left( V_i^{2-n} - V^{2-n} \right) \] (7)

**Velocity:** The expression for velocity in terms of altitude, is obtained from (2), (3) and the density-altitude relation
\[ V = \left( -\frac{2m K_2}{\rho_0 C_D A} \right)^{1/(2-n)} e^{-\beta y/(2-n)} \] (8)

**Angle of inclination:** From relation \( \sin \theta = -\frac{dy}{ds} \) and (5) and (7), the entry angle is given by
\[ \sin \theta = -\frac{K_2 (2-n)}{\beta V^{2-n}} \] (9)

**Lift-drag ratio:** From (1), (7) and (9) on simplification,
\[ \frac{L}{D} = \frac{(2-n)^2 K_2}{\beta} \left[ V^{4-2n} \right] \left\{ \frac{K_2 (2-n)}{\beta} \right\}^{2} - 1/2 \]
\[ -\frac{g}{K_2} \left( V^{4-2n} \right) \left\{ \frac{K_2 (2-n)}{\beta} \right\}^{2} \] (10)
where
\[ Z = \frac{1}{V^2} - \frac{1}{g R_0} \] (11)

**Range:** In the relation \( dR = \cos \theta ds \), substituting value of \( \cos \theta \) from (9) and \( ds \) from (7) on integrating and simplifying
\[ R = \frac{1}{\beta} \left[ \left\{ \frac{\beta V^{2-n}}{(2-n) K_2} \right\}^{2} - 1 \right] \frac{1}{4} - \left( \left\{ \frac{\beta V^{2-n}}{(2-n) K_2} \right\}^{2} - 1 \right) \frac{1}{4} \]
\[ -\frac{1}{\beta} \left[ \sec^{-1} \frac{\beta V^{2-n}}{(2-n) K_2} - \sec^{-1} \frac{\beta V^{2-n}}{(2-n) K_2} \right] \] (12)
Gurtu: Flight-Path Characteristics

It may perhaps be mentioned here that the expression for range as given by Loh is erroneous. The actual expression can be obtained by putting \( n = 0 \) in above equation.

When the index \( n \) vanishes, equations (4) to (10) give flight-path characteristics for constant deceleration flight.

**Rate of Change of Altitude varies as the \( p \)th Power of the Velocity**

The trajectory characteristics are now analysed for entry at

\[
\frac{dy}{dt} = K_3 V^p
\]

**Angle of inclination:** From the relation \( \sin \theta = -\frac{dy}{ds} \) and above relation,

\[
\sin \theta = -\left( \frac{C_{19} C_D A}{2 \beta m} \right) V^{-1+p}
\]

where

\[
C_{19} = 2 m \beta K_3 C_D A
\]

**Density:** Using (2) and (13), the relation \( d\rho = -\beta \rho dy \) and integrating

\[
\rho = C_{19} - \frac{C_{19}}{(1-p)} V^{-1+p}
\]

where

\[
C_{18} = \rho_1 + \frac{2 m \beta K_3}{C_D A (1-p) V^{1-p}}
\]

**Flight-path distance:** Substituting (16) into (2), integrating, and simplifying

\[
s = \left[ \frac{2 m}{C_{18} C_D A (1-p)} \right] \ln \left( \frac{(1-p) C_{18} V^{1-p} - C_{19}}{(1-p) C_{18} V^{1-p} - C_{19}} \right)
\]

**Range:** In the relation \( dR = \cos \theta \ ds \) substituting value of \( \cos \theta \) from (14) and \( ds \) from equation (18) and integrating

\[
R = \frac{2 m (1-p)}{C_D A} \int_V \left[ V^{2-2p} - \left( \frac{C_{19} C_D A}{2 \beta m} \right)^2 \right]^{\frac{1}{2}} \cdot
\]

\[
\left[ C_{19} V - C_{19} V^{2-p} (1-p) \right]^{-1} dV
\]

The value of \( R \) can be evaluated by numerical integration.

**Altitude:** Using relation \( \rho = \rho_0 e^{-\beta y} \) and (16) on simplification

\[
y = -\frac{1}{\beta} \ln \left[ e^{-\beta y_i} + \left\{ \frac{C_{19}}{(1-p) \rho_0} \right\} \left\{ V_i^{-1+p} - V^{-1+p} \right\} \right]
\]

**Velocity:** The expression for velocity, as a function of altitude, follows from above

\[
V^{-1+p} = V_i^{-1+p} - \left\{ \frac{(1-p) \rho_0}{C_{19}} \right\} \left\{ e^{-\beta y} - e^{-\beta y_i} \right\}
\]

**Deceleration and Maximum Deceleration:** The deceleration expression follows from (2) and (16) as

\[
\frac{dV}{dt} = -\left( \frac{C_D A}{2 m} \right) \left[ C_{18} V^2 - \frac{C_{19}}{(1-p)} V^{1+p} \right]
\]
The expression for maximum acceleration is
\[
\left( \frac{dV}{dt} \right)_{\text{max}} = \left( \frac{C_D A}{2 m} \right) \frac{C_{19}^{2/(1 - p)}}{2^{2/(1 - p)} C_{19}^{1 + p}/(1 - p)} \left( \frac{1 + p}{1 - p} \right)^{(1 + p)/(1 - p)}
\]  
(23)

where
\[
V_{(dV/dt)}_{\text{max}} = \left[ \left( \frac{C_{19}}{2 C_{19}} \right) \left( \frac{1 + p}{1 - p} \right) \right]^{1/(1 - p)}
\]  
(24)

**Time of flight:**—The expression for flight-time using (18) is
\[
t = \int \frac{ds}{V} = - \left( \frac{2 m}{C_D A} \right) \int \frac{p}{C_{19} V^2 - C_{19} V^1 + p} (1 - p) dV
\]  
(25)

which can be solved by numerical integration.

**Lift-drag ratio:**—Using equations (1), (14), (18), (22) and simplifying the lift-drag ratio is
\[
\frac{L}{D} = (1 - p) \left( \frac{2 g m}{C_D A} \right) \left[ \frac{V_{2 - 3 p} - K_s - K_s^2}{Z} \right] \frac{(1 - p)}{C_{19} (1 - p)} V^2 - C_{19} V^1 + p
\]  
(26)

As before it can be easily seen that if the exponent \( p \) is 0, equations (14) to (26) yield results for constant rate of descent flight.

**Constant Rate of Change of Range Flight.**

It is now proposed to analyze a different type of re-entry trajectory for which
\[
\frac{dR}{dt} = K_1
\]  
(27)

**Angle of inclination:**—The expression for angle of inclination from above is
\[
\sin \theta = (V^2 - K_s^2)^{1/2} V^{-1}
\]  
(28)

**Density:**—Using relations (2), (28), the relations \( d \rho = \beta \rho dy \) and \( \sin \theta = -dy/ds \) and simplifying
\[
\beta = C_{24} - C_{28} \left[ \ln \left\{ \frac{V + (V^2 - K_s^2)^{1/2}}{K_1} \right\} - \frac{(V^2 - K_s^2)^{1/2}}{V} \right],
\]  
(29)

where
\[
C_{24} = \rho \cdot + \quad C_{28} \left[ \ln \left\{ \frac{V_1 + (V_{2 - 3 p} - K_s^2)^{1/2}}{K_1} \right\} - \frac{(V_{2 - 3 p} - K_s^2)^{1/2}}{V_1} \right]
\]  
(30)

and
\[
C_{23} = 2 \beta m / C_D A
\]  
(31)

**Altitude:**—Using the altitude-density relation, (29) and simplifying, the altitude in terms of \( V \) is
\[
y = \frac{1}{\beta} \ln \left[ e^{-\beta y_i} - \frac{C_{29}}{\rho_0} \ln \left\{ \frac{V + (V^2 - K_i + K_f) \frac{V}{V_i}}{V_i + (V^2 - K_i + K_f) \frac{V}{V_i}} \right\} \right] + \frac{C_{23}}{\rho_0} \ln \left\{ \frac{(V^2 - K_i + K_f) \frac{V}{V_i}}{V} - \frac{(V^2 - K_i + K_f) \frac{V}{V_i}}{V_i} \right\}
\]
\[
\frac{dV}{dt} = \left( - \frac{C_D A}{2 m} \right) \left[ C_{24} - C_{23} \ln \left\{ \frac{V + (V^2 - K_i + K_f) \frac{V}{K_i}}{K_i} \right\} + C_{23} \frac{(V^2 - K_i + K_f) \frac{V}{V_i}}{V} \right] V^3
\]

Deceleration:—From (2) and (29) the expression for deceleration is

\[
\frac{dV}{dt} = \left( - \frac{C_D A}{2 m} \right) \left[ C_{24} - C_{23} \ln \left\{ \frac{V + (V^2 - K_i + K_f) \frac{V}{K_i}}{K_i} \right\} + C_{23} \frac{(V^2 - K_i + K_f) \frac{V}{V_i}}{V} \right] V^3
\]

For maximum deceleration

\[
\frac{d}{dV} \left( \frac{dV}{dt} \right) = 0
\]

which gives the value of \( V \) in implicit form as

\[
\frac{2 C_{24}}{C_{23}} = 2 \ln \left\{ \frac{V}{K_i} + \left[ \left( \frac{V}{K_i} \right)^2 - 1 \right]^{\frac{1}{2}} \right\} - 1 - \left( \frac{K_i}{V} \right)^2
\]

This value of \( V \) when substituted in (33) gives maximum deceleration.

Flight-path distance:—This follows from (2) and (29) on integration

\[
s = \left( - \frac{2 m}{C_D A} \right) \int_{V_i}^{V} \frac{dV}{V f(V)}
\]

where

\[
\rho = f(V) = C_{24} - C_{23} \ln \left\{ \frac{V + (V^2 - K_i + K_f) \frac{V}{K_i}}{K_i} \right\} + C_{23} \frac{(V^2 - K_i + K_f) \frac{V}{V_i}}{V}
\]

Range:—From the relation \( dR = \cos \theta \, ds \) and (27) and (36), on integration

\[
R = \left( - \frac{2 m K_i}{C_D A} \right) \int_{V_i}^{V} \frac{dV}{V^2 f(V)}
\]

Time of flight:—The expression for time of flight is

\[
t = \int_{V_i}^{V} \frac{ds}{V} = \left( - \frac{2 m}{C_D A} \right) \int_{V_i}^{V} \frac{dV}{V^2 f(V)}
\]

The last three characteristics, viz flight-path distance, range and time of flight can be evaluated by numerical integration.

CONCLUSIONS

The first entry case at \( \frac{dV}{dt} = K_i V^2 \) is particularly interesting. At \( n = 0 \) it reduces to the constant deceleration flight\(^5\). It may be mentioned that constant deceleration flight is specially useful for proper entry into Jupiter.

Further, the results for constant rate of average heat input flight\(^5\) can be deduced by putting \( n = -1 \) and suitably choosing \( K_i \).

The second and third entry cases, however, are more of general interest.
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REFERENCES