SOME RECURRANCE FORMULAE FOR G-FUNCTION OF TWO VARIABLES—II

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The object of this paper is to establish more recurrence relations for \(G\)-function of two variables. Certain known results for Meijer's \(G\)-function have been shown as particular cases.

Some identities and recurrence relations were recently given by the author as particular cases of finite series\(^1\) by using the derivatives of \(G\)-function of two variables\(^2,3\). The symbol \((k, \delta)\) represents the set of parameters \(k/\delta, (k+1)/\delta, \ldots, (k+\delta-1)/\delta\), where \(\delta\) is a positive integer and \((a_\alpha)\) stands for \(a_1, a_2, \ldots, a_\alpha\) throughout this paper.

The \(G\)-function of two variables defined by Agarwalla\(^4\) and Sharma\(^5\) has been denoted by Bajpai\(^6\) as

\[
G^{(m_1, m_2); (n_1, n_2), n_3} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} (a_\alpha); (c_\alpha) \\ (e_\alpha) \end{bmatrix} = \frac{1}{(2\pi i)^2} \int \prod_{j=1}^{m_1} \frac{\Gamma(b_j-s)}{\Gamma(1-b_j+s)} \prod_{j=1}^{n_1} \Gamma(1-a_j+s) \prod_{j=1}^{m_2} \frac{\Gamma(d_j-t)}{\Gamma(1-d_j+t)} \prod_{j=1}^{n_2} \Gamma(1-c_j+t) \prod_{j=1}^{n_3} \Gamma(1-e_j+s+t)
\]

\[
\int_{L_1} \prod_{j=m_1+1}^{n_1} \frac{\Gamma(1-a_j+s)}{\Gamma(a_j-s)} \prod_{j=m_2+1}^{n_2} \frac{\Gamma(1-c_j+t)}{\Gamma(c_j-t)} \prod_{j=m_3+1}^{n_3} \frac{\Gamma(1-e_j+s+t)}{\Gamma(1-f_j+s+t)}
\]

\[
x^s y^t ds \, dt.
\]

The contour \(L_1\) is in the \(s\)-plane and runs from \(-i\infty\) to \(+i\infty\) with loops if necessary, to ensure that the poles of \(\Gamma(b_j-s), j=1, 2, \ldots, m_1\) lie on the right and the poles of \(\Gamma(1-a_j+s), j=1, 2, \ldots, n_1\) and \(\Gamma(1-e_j+s+t), j=1, 2, \ldots, n_3\) to the left of the contour. Similarly the contour \(L_2\) is in the \(t\)-plane and runs from \(-i\infty\) to \(+i\infty\) with loops if necessary, to ensure that the poles of \(\Gamma(d_j-t), j=1, 2, \ldots, m_2\) lie on the right and the poles of \(\Gamma(1-c_j+t), j=1, 2, \ldots, n_2\) and \(\Gamma(1-e_j+s+t), j=1, 2, \ldots, n_3\) on the left of the contour.

Provided that

\[
0 < m_1 \leq q_1, \quad 0 < m_2 \leq q_2, \quad 0 < n_1 \leq p_1, \quad 0 < n_2 \leq p_2, \quad 0 < n_3 \leq p_3
\]

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the integral converges if
\[
(p_3+q_3+p_1+q_1) < 2 (m_1+n_1+n_3); (p_3+q_3+p_2+q_2) < 2 (m_2+n_2+n_3) \\
| \arg x | < [m_1+n_1+n_3-\frac{1}{2} (p_3+q_3+p_1+q_1)] \pi, \\
| \arg y | < [m_2+n_2+n_3-\frac{1}{2} (p_3+q_3+p_2+q_2)] \pi.
\]

The right hand side of (1) shall henceforth be denoted by \( G \left[ \begin{array}{c} x \\ y \end{array} \right] \), whenever there is no chance of misunderstanding, and is the required \( G \)-function of two variables.

We establish the following identities:

If one value of \( a_h, h=1, 2, \ldots, n_1 \) is equal to one value of \( b_j, j=m_1+1, \ldots, q_1 \); one value of \( c_h, h=1, 2, \ldots, n_2 \) is equal to one value of \( d_j, j=m_2+1, \ldots, q_2 \) and one value of \( e_h, h=1, 2, \ldots, n_3 \) is equal to one value of \( f_j, j=1, 2, \ldots, q_3 \), the \( G \)-function of two variables reduces to one of lower order. For example

\[
G \left[ \begin{array}{c} (a_{p_1}); (c_{p_1}) \\ (e_{p_1}) \\ (b_1); d_1, \ldots, d_{q_1}-1, a_1; (d_{q_1}) \\ (f_{q_1}) \end{array} \right] = G \left[ \begin{array}{c} a_2, \ldots, a_{p_1}; (c_{p_1}) \\ (e_{p_1}) \\ b_1, \ldots, b_{q_1}-1 (d_{q_1}) \end{array} \right] 
\]

(2)

\[
G \left[ \begin{array}{c} x \\ y \end{array} \right] = G \left[ \begin{array}{c} x \\ y \end{array} \right]
\]

(3)

\[
G \left[ \begin{array}{c} x (a_{p_1}); (c_{p_1}) \\ (e_{p_1}) \\ (b_{q_1}); d_{q_1}, \ldots, q_3 \\ e_1, f_2, f_3, \ldots, f_{q_3} \end{array} \right] = G \left[ \begin{array}{c} x (a_{p_1}); (c_{p_1}) \\ (e_{p_1}) \\ (b_{q_1}); (d_{q_1}) \\ f_2, \ldots, f_{q_3} \end{array} \right]
\]

(4)

\[
G \left[ \begin{array}{c} x \\ y \end{array} \right] = G \left[ \begin{array}{c} x \\ y \end{array} \right]
\]

(5)

Also, if one value of \( a_h, h=n_1+1, \ldots, p_1 \) is equal to one value of \( b_j, j=1, 2, \ldots, m_1 \); one value of \( c_h, h=n_2+1, \ldots, p_2 \) is equal to one value of \( d_j, j=1, 2, \ldots, m_2 \), then the \( G \)-function of two variables reduces to one of a lower order. For example
The roof of the above-mentioned iderlitiqies are very simple and are therefore omitted.

Proof

To prove (B), expressing G-function on the left hand side as (1) and replacing a by ks and t by kt, we get

\[
G \left[ \begin{array}{l} x \\ y \\
\end{array} \right] = \left( 2\pi \right)^{u} \delta_{k} G \left[ \begin{array}{l} \frac{x^{i}}{\left( k^{i} - s + x \right)} \\
\frac{y^{j}}{\left( k^{j} + s - x \right)} \\
\end{array} \right]
\]

where

\[
u = \frac{1}{2} \left( p_{1} + q_{1} + p_{2} + q_{2} + p_{3} + q_{3} - (n_{1} + n_{1} + n_{2} + n_{3}) \right)
\]

Proof

To prove (9), expressing G-function on the left hand side as (1) and replacing s by ks and t by kt, we get

\[
\frac{1}{\left( 2\pi \right)^{4}} \int_{L_{1}}^{L_{2}} \int_{L_{1}}^{L_{2}} \prod_{j=1}^{m_{1}} \Gamma \left( b_{j} - ks \right) \prod_{j=1}^{m_{2}} \Gamma \left( 1 - a_{j} + ks \right) \prod_{j=1}^{m_{3}} \Gamma \left( d_{j} + kt \right) \prod_{j=1}^{m_{4}} \Gamma \left( 1 - c_{j} + kt \right)
\]

\[
\int_{L_{1}}^{L_{2}} \int_{L_{1}}^{L_{2}} \prod_{j=m_{1}+1}^{n_{1}} \Gamma \left( 1 - e_{j} + ks + kt \right) \prod_{j=m_{2}+1}^{n_{2}} \Gamma \left( 1 - f_{j} + ks + kt \right)
\]

\[
\prod_{j=m_{3}+1}^{n_{3}} \Gamma \left( e_{j} - ks - kt \right) \prod_{j=m_{4}+1}^{n_{4}} \Gamma \left( 1 - f_{j} + ks + kt \right)
\]

\[
\prod_{j=1}^{n_{1}} \Gamma \left( 1 - e_{j} + ks + kt \right) \prod_{j=1}^{n_{2}} \Gamma \left( 1 - f_{j} + ks + kt \right)
\]

\[
\prod_{j=1}^{n_{3}} \Gamma \left( e_{j} - ks - kt \right) \prod_{j=1}^{n_{4}} \Gamma \left( 1 - f_{j} + ks + kt \right)
\]
Now using multiplication formula for Gamma function \([7, \text{p. 11, (1)}]\) and (1), the formula (9) is established.

\[
G \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{1}{2\pi i} \left\{ (e)^{i\pi b}m_{n+1} \right\} G \begin{bmatrix} (m_1 + 1, m_2); (n_1, n_2), n_3 \\ (p_1, p_2), p_3; (q_1, q_2), q_3 \end{bmatrix} \begin{bmatrix} xe^{-i\pi} (a_p); (c_p) \\ ye^{-i\pi} (b_q); (d_q) \end{bmatrix}
\]

\[-(e^{-i\pi b}m_{n+1}) G \begin{bmatrix} (m_1 + 1, m_2); (n_1, n_2), n_3 \\ (p_1, p_2), p_3; (q_1, q_2), q_3 \end{bmatrix} \begin{bmatrix} xe^{i\pi} (a_p); (c_p) \\ ye^{i\pi} (b_q); (d_q) \end{bmatrix}
\]

\[
G \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{1}{2\pi i} \left\{ (e)^{i\pi a}n_{n+1} \right\} G \begin{bmatrix} (m_1, m_2); (n_1 + 1, n_2), n_3 \\ (p_1, p_2), p_3; (q_1, q_2), q_3 \end{bmatrix} \begin{bmatrix} xe^{-i\pi} (a_p); (c_p) \\ ye^{-i\pi} (b_q); (d_q) \end{bmatrix}
\]

\[-(e^{-i\pi a}n_{n+1}) G \begin{bmatrix} (m_1, m_2); (n_1 + 1, n_2), n_3 \\ (p_1, p_2), p_3; (q_1, q_2), q_3 \end{bmatrix} \begin{bmatrix} xe^{i\pi} (a_p); (c_p) \\ ye^{i\pi} (b_q); (d_q) \end{bmatrix}
\]

\[
G \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{1}{2\pi i} \left\{ (e)^{i\pi e}n_{n+1} \right\} G \begin{bmatrix} (m_1, m_2); (n_1, n_2 + 1), n_3 \\ (p_1, p_2), p_3; (q_1, q_2), q_3 \end{bmatrix} \begin{bmatrix} xe^{-i\pi} (a_p); (c_p) \\ ye^{-i\pi} (b_q); (d_q) \end{bmatrix}
\]

\[-(e^{-i\pi e}n_{n+1}) G \begin{bmatrix} (m_1, m_2); (n_1, n_2 + 1), n_3 \\ (p_1, p_2), p_3; (q_1, q_2), q_3 \end{bmatrix} \begin{bmatrix} xe^{i\pi} (a_p); (c_p) \\ ye^{i\pi} (b_q); (d_q) \end{bmatrix}
\]

\[
G \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{1}{2\pi i} \left\{ (e)^{i\pi c}n_{n+1} \right\} G \begin{bmatrix} (m_1, m_2); (n_1, n_2), n_3 + 1 \\ (p_1, p_2), p_3; (q_1, q_2), q_3 \end{bmatrix} \begin{bmatrix} xe^{-i\pi} (a_p); (c_p) \\ ye^{-i\pi} (b_q); (d_q) \end{bmatrix}
\]

\[-(e^{-i\pi c}n_{n+1}) G \begin{bmatrix} (m_1, m_2); (n_1, n_2), n_3 + 1 \\ (p_1, p_2), p_3; (q_1, q_2), q_3 \end{bmatrix} \begin{bmatrix} xe^{i\pi} (a_p); (c_p) \\ ye^{i\pi} (b_q); (d_q) \end{bmatrix}
\]
Proof

To prove (10), expressing the $G$-function on the left hand side as (1) and multiplying the numerator and denominator by $\Gamma(b_{m+1}+s)$, we get,

$$\frac{1}{(2\pi i)^2} \left\{ \sum_{j=1}^{m+1} \frac{\Pi_{j=1}^{n_1} \Gamma(b_j-s) \Pi_{j=1}^{n_2} \Gamma(d_j-t) \Pi_{j=1}^{n_3} \Gamma(1-c_j+s+t)}{\Pi_{j=1}^{n_4} \Gamma(1-e_j+s+t)} \right\}.$$

Now by virtue of the relation

$$\Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z} = \frac{2\pi i}{e^{i\pi z} - e^{-i\pi z}}$$

we see that

$$\Gamma(b_{m+1}+s) \Gamma(1-b_{m+1}+s) = \frac{2\pi i}{(e)^{i\pi (b_{m+1}+s)} - (e)^{-i\pi (b_{m+1}+s)}} \tag{16}$$

The relation (10) is proved by using (15), (16) and (1).

By adopting the same procedure as for (10), the formulae (11)–(14) are proved.

$$G = \sum_{(m_1 + 1, m_2 + 1): (n_1, n_2), n_3} \left\{ \begin{array}{c} x \mid a, (a_p) ; c, (c_p) \\ y \mid b, (b_q) ; d, (d_q) \end{array} \right\} = (-1)^r (-1)^o \Gamma \left[ \begin{array}{c} x \\ y \end{array} \right] \tag{17}$$

where $a_n = a + r$, $c_n = c + k$, $r$ and $k$ are integers.

$$G = \sum_{(m_1, m_2); (n_1 + 1, n_2 + 1), n_3} \left\{ \begin{array}{c} x \mid a, (a_p) ; c, (c_p) \\ y \mid b, (b_q) ; d, (d_q) \end{array} \right\} = (-1)^r (-1)^o \Gamma \left[ \begin{array}{c} x \\ y \end{array} \right] \tag{18}$$

where $a-b = r$, $c-d = k$, $r$ and $k$ are integers or zero.

(17) and (18) are proved by using (1) and Rainville\(^8\)

$$x^n \frac{\partial^n G}{\partial x^n} \left[ \begin{array}{c} x \\ y \end{array} \right] = \Gamma \left[ \begin{array}{c} m_1, m_2; (n_1 + 1, n_2), n_3 \\ p_1 + 1, p_2 + 1, p_3; (q_1 + 1, q_2 + 1), q_3 \end{array} \right] \tag{19}$$
A similar result is true for
\[ y^n \frac{\partial^n}{\partial y^n} G \left[ \frac{x}{y} \right] \]
\[ x^n \frac{n}{\partial x^n} G \left[ \frac{x^{-1}}{y} \right] = \left( - \right)^n G \left[ \frac{(m_1, m_2); (n_1 + 1, n_2), n_3}{(p_1 + 1, p_2), p_3; (q_1 + 1, q_2), q_3} \right] \]
\[ x^{-1} \left[ 1 - n_1 \left( a_{p_3}; c_{p_3} \right) \right] \]
\[ (e_{p_3}) \left[ (b_{q_3}, 1; d_{q_3}) \right] \]

(20)

A similar result holds for
\[ y^n \frac{\partial^n}{\partial y^n} G \left[ \frac{x}{y^{-1}} \right] \]

The proofs for (19) and (20) are very simple and follow by expressing the G-function on the left hand side as in (1), changing the order of integration and differentiation and again using (1).

**Particular cases:**—Putting \( m_2 = q_2 = 1 \), \( n_2 = n_3 = p_2 = p_3 = q_3 = 0 \) and making use of the formula given by Bajpai's viz.

\[ G \left[ \frac{(m, 1); (n, 0), 0}{(p, 0), 0; (q, 1), 0} \right] = e^{-y} G \left[ \frac{(a_p, 0); \infty}{(b_q, 0) \left[ (p, q) \right]} \right] \]

we get the known results from (2), (6), (10), (12), (19) and (20).

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**REFERENCES**

1. **Gulati, H. C.,** Finite series for G-function of two variables (Communicated for publication).
3. **Gulati, H. C.,** Derivatives of G-function of two variables (Communicated for publication).
6. **Bajpai, S. D.,** Some results involving G-function of two variables (Communicated for publication).