ROCKET LOCATIONS FOR MINIMUM TIME INTERCEPTION

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Interception of target rocket moving along a trajectory in a gravitational field by an interceptor rocket has been analysed to achieve interception in minimum time duration. Numerical values of the parameters of the intercepting trajectory and locations of the target rocket for launch and target orbits in Earth's gravitational field have been calculated for different impulsive velocity changes applied to the interceptor rocket.

In reference (1) has been studied the interception of moving rockets at preassigned destinations in gravitational field. In that the emphasis has been laid on the fuel minimization in the interception operation. However, conditions may arise when interception in minimum flight time of interceptor rocket may become necessary. The problem of interception of a moving target rocket by an interceptor rocket in minimum time with specified fuel expenditure in the interception operation is analysed in this paper. The paths of both the target and interceptor rockets are supposed to be known. The location of the target rocket at the instant the interception rocket is fired so as to hit the former in minimum time with given impulsive velocity change imparted to the interceptor rocket has been found out. The numerical results have been calculated for different impulsive velocity changes.

INTERCEPTION TRAJECTORY PARAMETERS

Let the equations of the paths of the interceptor and target rockets be

\[ \frac{l_1}{r} = 1 + e_1 \cos \theta \]  
\[ \frac{l_2}{r} = 1 + e_2 \cos (\theta - \xi) \]

where \( l, e \) and \( \xi \) denote latus rectum, eccentricity, angle between the major axes of the two orbits respectively and suffixes 1 and 2 correspond to the paths of interceptor and target rockets (hereafter called launch and destination orbits). \( r \) and \( \theta \) represent polar coordinates with force centre as pole and line drawn from the force centre to the perigee of the launch orbit as initial line. Let \((r_1, \theta_1)\) be the given launch point (Fig 1) at which the interceptor rocket is fired and \((r_2, \theta_2)\) the destination point where the target is hit by the interceptor rocket. Suppose \( \theta_2 = \theta_1 + \phi \) where \( \phi \) is the transfer angle. Now if \( \Delta V \) is the impulsive velocity change applied to the interceptor rocket, we have

\[ V_i^2 = V_1^2 + (\Delta V)^2 + 2\Delta V \cdot V_1 \cos \alpha \]  

Fig. 1—Geometry of interception.
where \( V \) represents velocity and suffix \( i \) denotes values at the launch point just after application of the impulse and \( \alpha \) is the angle between \( V_1 \) and \( \Delta V \). If \( \gamma \) is the angle between velocity vector and local horizontal, \( \alpha \) can be expressed as

\[
\alpha = (\gamma_i - \gamma_1) + \sin^{-1} \left[ \frac{V_1}{\Delta V} \varepsilon \sin (\gamma_i - \gamma_1) \right]
\]

where \( \gamma_1 \) and \( V_1 \) are given by

\[
\tan \gamma_1 = \frac{e_i \sin \theta_1}{1 + e_i \cos \theta_1}
\]

\[
V_1 = \left[ \frac{K}{l_1} (1 + e_i^2 + 2e_i \cos \theta_1) \right]^{\frac{1}{2}}
\]

\( K \) being the gravitational parameter. With the help of (4) to (6), (3) shows that for specified \( \Delta V \); \( V_i \) is a function of \( \gamma_i \).

If \( E \) and \( a \) are the eccentricity and semi-major axis of the elliptic interception trajectory, we have

\[
E = \left[ \left( \frac{r_i V_i^2}{K} - 1 \right) \right]^{\frac{1}{2}} \cos^2 \gamma_i + \sin^2 \gamma_i
\]

\[
a = \frac{r_i}{\left( 2 - \frac{r_i V_i^2}{K} \right)}
\]

Then flight time \( T \) of the interceptor rocket will be given² by

\[
T = \left( \frac{a^3}{K} \right)^{\frac{1}{2}} \left[ (\psi_2 - \psi_1) - E (\sin \psi_2 - \sin \psi_1) \right]
\]

where \( \psi_1 \) and \( \psi_2 \) are the eccentric anomalies of the launch and destination points corresponding to the interception trajectory and

\[
\cos \psi = \frac{a - r}{aE}
\]

Therefore \( T \) can be written as

\[
T = \left( \frac{a^3}{K} \right)^{\frac{1}{2}} \left[ \cos^{-1} \left( \frac{a - r_2}{aE} \right) - \cos^{-1} \left( \frac{a - r_1}{aE} \right) - E \left( \sin \left\{ \cos^{-1} \left( \frac{a - r_2}{aE} \right) \right\} - \sin \left\{ \cos^{-1} \left( \frac{a - r_1}{aE} \right) \right\} \right) \right]
\]

where \( r_2 \) is given by (2) as

\[
r_2 = \frac{l_2}{1 + e_2 \cos (\theta_1 + \phi - \xi)}
\]

**MINIMUM INTERCEPTION TIME**

Since interception trajectory passes through the points \((r_1, \theta_1)\) and \((r_2, \theta_2)\) we have¹

\[
\frac{r_i V_i^2}{K} = \frac{r_1}{r_2} - \cos \phi + \sin \phi \tan \gamma_i
\]
Substitution of the value of $r_2$ in (10), rearrangement of the terms and simplification gives

$$(1 + C_1 C_2) \cos \phi + C_1 C_4 \sin \phi + C_1 C_2 - 1 = 0$$

(11)

where

$$C_1 = \frac{r_1 V_1^2 \cos^2 \gamma_i}{K}, \quad C_2 = \frac{r_1}{l_2}, \quad C_3 = \frac{r_1 e_2 \cos (\theta_1 - \xi)}{l_2} - 1$$

and

$$C_4 = \tan \gamma_i - \frac{r_1 e_2 \sin (\theta_1 - \xi)}{l_2}$$

From equation (11) we obtain

$$\tan \frac{\phi}{2} = -\frac{C_1 C_4 \pm [(C_1 C_4^2 - C_1 (C_2 + C_3) \{ C_1 (C_2 - C_3) - 2 \}]^{\frac{1}{2}}}{C_1 (C_2 - C_3) - 2}$$

(12)

Positive sign is taken before the radical because negative sign will correspond to greater value of $\phi$ for the interception trajectory and therefore greater interception time. With (8) to (8) and (10), interception time $T$ in (9) is a function of single variable $\gamma_i$ and therefore for minimum time interception

$$\frac{3T}{3\gamma_i} = \frac{3T}{3a} \cdot \frac{3a}{3\gamma_i} + \frac{3T}{3\phi} \cdot \frac{3\phi}{3\gamma_i} + \frac{3T}{3E} \cdot \frac{3E}{3\gamma_i} = 0$$

(13)

where

$$\frac{3T}{3\phi} = \left( \frac{a}{K} \right)^{\frac{3}{2}} \frac{e_2 \sin (\phi + \theta_1 - \xi) r_3}{l_2 (\alpha E)^2 - (a - r_3)^2}$$

$$\frac{3T}{3E} = \left( \frac{a}{K} \right)^{\frac{3}{2}} \frac{1}{E} \left\{ \frac{r_2 (a - r_2)}{(\alpha E)^2 - (a - r_2)^2} \right\}^{\frac{1}{2}} - \frac{\frac{1}{E}}{(\alpha E)^2 - (a - r_1)^2}$$

$$\frac{3a}{3\gamma_i} = \frac{a M a^2}{K}$$

$$\frac{3E}{3\gamma_i} = \frac{\cos \gamma_i}{E} \left\{ \sin \gamma_i \left\{ 1 - \left( \frac{r_1 V_1^2}{K} - 1 \right)^2 \right\} + \frac{r_1}{K} \frac{M \cos \gamma_i}{(r_1 V_1^2 - 1)} \right\} \sin \theta$$

and

$$M = -2\Delta V V_1 \left[ 1 + \frac{V_1 \cos (\gamma_i - \gamma_i)}{(\Delta V)^2 - \{ V_1 \sin (\gamma_i - \gamma_i) \}^2} \right] \sin \theta$$
Equation (13) can be numerically solved which will give $\gamma_i$ and then $E$, $a$ and $\phi$ can be obtained from equations (7), (8) and (12) respectively. Having known $E$, $a$ and $\phi$, interception time for minimum time interception trajectory can be evaluated from (9) and corresponding $\theta_2$ also becomes known.

**TARGET ROCKET LOCATIONS**

Let $(r_d, \theta_d)$ be the position of the target rocket on the destination orbit at the instant the interceptor rocket is fired from the launch point so as to intercept the former at the point $(r_2, \theta_2)$ the destination point corresponding to the minimum time interception trajectory. Then

$$
\frac{1}{\sqrt{K}} \left( \frac{l_2}{1 - e_2^2} \right)^{\frac{3}{2}} \left[ 2 \left\{ \tan^{-1} \left( \left\{ \frac{1 - e_2}{1 + e_2} \right\}^\frac{1}{2} \tan \frac{\theta_2 - \xi}{2} \right) - \tan^{-1} \left( \left\{ \frac{1 - e_2}{1 + e_2} \right\}^\frac{1}{2} \tan \frac{\theta_d - \xi}{2} \right) \right\},
\tan \left( \frac{\theta_d - \xi}{2} \right) \right] - e_2 \left( 1 - e_2^2 \right)^\frac{1}{2} \left\{ \frac{\sin (\theta_2 - \xi)}{1 + e_2 \cos (\theta_2 - \xi)} - \frac{\sin (\theta_d - \xi)}{1 + e_2 \cos (\theta_d - \xi)} \right\} = T \tag{14}
$$

From (14), the unknown $\theta_d$ can be evaluated and then from (2) $r_d$ can be found out. $(r_d, \theta_d)$ will give the location of the target rocket for minimum time interception. The equations have been numerically solved for two arbitrarily chosen launch and destination orbits in Earth's gravitational field to obtain rocket locations for minimum time interception and parameters of the corresponding interception trajectories for different specified values of $\Delta V$. The results obtained are given in Table 1. The launch orbit is specified by $l_1 = 4243$ miles, $e_1 = 0.04$ and the destination orbit by $l_2 = 6006$ miles, $e_2 = 0.3$ and $\xi = 30^\circ$. The vectorial $l_\theta$ of the launch point is taken equal to $60^\circ$. A study of Table 1 shows that as expected with increasing velocity impulse $\Delta V$, minimum interception time decreases.

**TABLE 1**

<table>
<thead>
<tr>
<th>$\Delta V$ (miles/sec)</th>
<th>$\gamma_i$</th>
<th>$T$ (sec)</th>
<th>$r_d$ (miles)</th>
<th>$\theta_d$</th>
<th>$E$</th>
<th>$a$ (miles)</th>
<th>$\theta_a$</th>
<th>$\xi$</th>
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<tbody>
<tr>
<td>1.00</td>
<td>7° 15'</td>
<td>1839.2</td>
<td>7035.73</td>
<td>149° 12'</td>
<td>0.437</td>
<td>7200.54</td>
<td>76° 30'</td>
<td>4978.00</td>
</tr>
<tr>
<td>1.25</td>
<td>11° 48'</td>
<td>1039.4</td>
<td>5928.93</td>
<td>117° 31'</td>
<td>0.457</td>
<td>7141.43</td>
<td>67° 2'</td>
<td>4845.56</td>
</tr>
<tr>
<td>1.50</td>
<td>15° 23'</td>
<td>706.2</td>
<td>5475.30</td>
<td>101° 9'</td>
<td>0.480</td>
<td>7107.13</td>
<td>63° 54'</td>
<td>4808.65</td>
</tr>
<tr>
<td>1.75</td>
<td>18° 44'</td>
<td>534.7</td>
<td>5266.14</td>
<td>91° 36'</td>
<td>0.502</td>
<td>7021.42</td>
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<tr>
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<td>431.7</td>
<td>5140.11</td>
<td>85° 50'</td>
<td>0.546</td>
<td>7365.63</td>
<td>61° 35'</td>
<td>4783.51</td>
</tr>
</tbody>
</table>

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**REFERENCES**