FOURIER SERIES FOR FOX’S H-FUNCTION

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Two integrals involving Fox’s H-function have been evaluated and used to establish two Fourier series for the H-function. On specialising the parameters, the H-function can be reduced to Meijer G-function, MacRobert’s E-function, generalised hypergeometric functions and many other higher transcendental functions. The results established are of a general character.

Carlson & Greinan have obtained a cosine series for Gegenbauer’s function. MacRobert has established a cosine and sine series for MacRobert’s E-function. Jain, Kesawan and Bajpai have obtained some Fourier series for Meijer’s G-function. Except a few all the Fourier series for the G-function have been established following the results of MacRobert. The Fourier series for H-function in this paper have been obtained with the help of a result given by Nielsen.

The H-function introduced by Fox is represented and defined as follows:

\[ H_{m,n}^{p,q} \left[ \begin{array}{c|c} (a_1, e_1), \ldots, (a_p, e_p) \\ (b_1, f_1), \ldots, (b_q, f_q) \end{array} \right] = \frac{1}{2\pi i} \left\{ \Pi_{j=1}^{m} \Gamma (b_j - f_j s) \frac{1}{j} \Pi_{j=1}^{n} \Gamma (1 - a_j + e_j s) z^s ds, \right. \]

where an empty product is interpreted as 1, 0 < m < q, 0 < n < p; e’s and f’s are all positive numbers, L is suitable contour of Barnes type such that the poles of \( \Gamma (b_j - f_j s) \), \( j = 1, \ldots, m \) lie on the right hand side of the contour and those of \( \Gamma (1 - a_j + e_j s) \), \( j = 1, \ldots, n \) lie on the left hand side of the contour.

Asymptotic expansion and analytic continuation of the H-function have been discussed by Braaksma.

Following formulae are required in the proofs:

(a) The following integrals:

\[ \int_0^{\pi} (\sin \theta)^\rho \cos \theta \, d\theta = \frac{\pi}{2} \frac{\Gamma (1 + \rho) \cos (\pi u/2)}{\Gamma \left( 1 + \frac{\rho + u}{2} \right) \Gamma \left( 1 + \frac{\rho - u}{2} \right)}. \]
\[ \int_0^\pi (\sin \theta)^\rho \sin u \theta \, d\theta = \frac{\pi \Gamma (1 + \rho) \sin (\pi u/2)}{2\rho \Gamma \left( 1 + \frac{\rho + u}{2} \right) \Gamma \left( 1 + \frac{\rho - u}{2} \right)}, \]

where \( \rho > -1 \)

(b) The duplication formula for the gamma-function:

\[ (\pi)^{\frac{1}{2}} \Gamma (2z) = 2^{(2z - 1)} \Gamma (z) \Gamma (z + \frac{1}{2}). \]

In what follows for sake of brevity \((a_1, e_1), \ldots, (a_p, e_p)\); \(a_\rho\) stand for \(a_1, \ldots, a_p\) and the symbol \(\triangle (\delta, \alpha)\) represents the set of parameters \((\alpha/\delta, \frac{\alpha + 1}{\delta}, \ldots, \frac{\alpha + \delta - 1}{\delta})\), where \(\delta\) is a positive integer.

**The Integrals**

The integrals to be established are:

\[ \int_0^\pi (\sin \theta)^\rho \sin u \theta \, d\theta \]

\[ = (\pi)^{\frac{1}{2}} \cos \frac{\pi u}{2} H_{p, q}^{m, n-2} \left[ \left( 1 + \frac{\rho}{2}, \delta \right), (-\rho/2, \delta), (a_p, e_p) \right] \]

\[ \int_0^\pi (\sin \theta)^\rho \sin u \theta \, d\theta \]

\[ = (\pi)^{\frac{1}{2}} \sin \frac{\pi u}{2} H_{p, q}^{m, n-2} \left[ \left( 1 - \frac{\rho}{2}, \delta \right), (-\rho/2, \delta), (a_p, e_p) \right] \]

where \(\delta\) is a positive number and

\[ \sum_{j=1}^p e_j - \sum_{j=1}^q f_j \leq 0, \quad \sum_{j=1}^n e_j - \sum_{j=n+1}^p e_j + \sum_{j=1}^m f_j - \sum_{j=m+1}^q f_j \equiv K > 0, \]

\[ |\arg z| < \frac{1}{2} K \pi, \quad Re 2 \delta b_j f_j > 1 - \rho \quad (j = 1, \ldots, m). \]
Proof

To prove (5), expressing the H-function in the integrand as a Mellin-Barnes type integral (1) and interchanging the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i} \int \prod_{j=1}^{m} \frac{\Gamma \left( b_j - f_j s \right)}{\prod_{j=1}^{q} \Gamma \left( 1 - b_j + f_j s \right)} \prod_{j=m+1}^{p} \frac{\Gamma \left( 1 - a_j + e_j s \right) z^s}{\Gamma \left( a_j - e_j s \right)} \left( \sin \theta \right)^{\rho + 2\delta s} \cos \theta \, d\theta \, ds.$$  

Now evaluating the inner integral with the help of (2) and using duplication formula for Gamma-function (4), we get

$$\frac{1}{2\pi i} \int \prod_{j=1}^{m} \frac{\Gamma \left( b_j - f_j s \right)}{\prod_{j=m+1}^{q} \Gamma \left( 1 - b_j + f_j s \right)} \prod_{j=n+1}^{p} \frac{\Gamma \left( 1 + \rho/2 + \delta s \right)}{\Gamma \left( 1 + \rho/2 + \delta s \right)} \cos \left( \pi u/2 \right) \times \frac{\Gamma \left( 1 + \rho/2 + \delta s \right)}{\Gamma \left( 1 + \rho/2 + \delta s \right)} \, ds.$$  

On applying (1), the result (5) is established.

The integral (6) is established on applying the same procedure and using (3).

**Fourier Series**

The Fourier series to be obtained are

$$(\sin \theta)^{\rho} H_{m, n}^{p, q} \left[ z (\sin \theta)^{2\delta} \right] = \frac{1}{\sqrt{\pi}} \frac{H_{m, n+1}^{p+1, q+1}}{\prod_{r=1}^{\infty} \left( \frac{1}{2} - \rho, \delta \right)} \left( \frac{\rho}{2}, \delta \right), (a_p, e_p)$$

$$+ \frac{2}{(\pi i)^{\rho}} \int \sum_{r=1}^{\infty} \left[ z \left( \left( \frac{1}{2} - \rho, \delta \right), (-\rho/2, \delta), (a_r, e_p) \right) ight] \times \cos \frac{\pi r}{2} \cos r\theta,$$  

$$\left( b_q, f_q \right), \left( -\frac{\rho + u}{2}, \delta \right), \left( -\frac{\rho - u}{2}, \delta \right)$$

(7)
\[
\begin{align*}
\sin \theta \rho H_{p, q}^{m, n} & \left[ z \left( \sin \theta \right)^{2} \begin{pmatrix} (a_p, e_p) \\ (b_q, f_q) \end{pmatrix} \right] \\
& = \frac{2}{(\pi)^{2}} \sum_{r=1}^{\infty} H_{p+2, q+2}^{m, n} \left[ z \left( \begin{pmatrix} \frac{1}{2} - \rho, \delta \\ -\rho/2, \delta \end{pmatrix}, (a_p, e_p) \right. \\
& \left. \left. \left( -\rho + u, \delta \right), \left(-\frac{\rho - u}{2}, \delta \right) \right) \right] \\
& \times \sin \frac{\pi r}{2} \sin r \theta,
\end{align*}
\]

where \( \delta \) is a positive number and

\[
\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j < 0, \quad \sum_{j=1}^{n} e_j - \sum_{j=m+1}^{n} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \equiv K > 0,
\]

\[| \arg z | = \frac{1}{2} K < \pi, \quad \text{Re } 2 \delta b_j f_j > 1 - \rho \quad (j = 1, \ldots, m), \quad 0 < \theta < \pi.
\]

**Proof**

To establish (7), let

\[
f(\theta) = (\sin \theta)^{2} H_{p, q}^{m, n} \left[ z \left( \sin \theta \right)^{2} \begin{pmatrix} (a_p, e_p) \\ (b_q, f_q) \end{pmatrix} \right] = \frac{C_0}{2} + \sum_{r=1}^{\infty} C_r \cos r \theta.
\]

Equation (9) is valid since \( f(\theta) \) is continuous and of bounded variation in the open interval \((0, \pi)\), when \( \rho > 0 \).

Multiplying both sides of (9) by \( \cos (u \theta) \) and integrating with respect to \( \theta \) from 0 to \( \pi \), we get

\[
\int_{0}^{\pi} (\sin \theta)^{2} \cos u \theta H_{p, q}^{m, n} \left[ z \left( \sin \theta \right)^{2} \begin{pmatrix} (a_p, e_p) \\ (b_q, f_q) \end{pmatrix} \right] d\theta
\]

\[
= \frac{C_0}{2} \int_{0}^{\pi} \cos u \theta d\theta + \sum_{r=1}^{\infty} C_r \int_{0}^{\pi} \cos r \theta \cos u \theta d\theta
\]

Now using (5) and the orthogonality property of cosine functions, we have

\[
C_u = \frac{2}{(\pi)^{2}} \cos \frac{\pi u}{2} \left( \begin{pmatrix} \frac{1}{2} - \rho, \delta \\ -\rho/2, \delta \end{pmatrix}, (a_p, e_p) \right. \\
& \left. \left. \left( -\rho + u, \delta \right), \left(-\frac{\rho - u}{2}, \delta \right) \right) \right] \\
& \times \sin \frac{\pi r}{2} \sin r \theta.
\]

From (9) and (10), the result (7) is obtained.

**To prove (8), let**
\[ f(\theta) = (\sin \theta)^p H_{p, q}^{m, n} \left[ z (\sin \theta)^{2\delta} \left| \begin{array}{c} (a_p, e_p) \\ (b_q, f_q) \end{array} \right| \right] = \sum_{r=1}^{\infty} C_r \sin \theta \]  

(11)

Multiplying both sides of (11) by \( \sin (u \theta) \) and integrating with respect to \( \theta \) from 0 to \( \pi \) then using (6) and the orthogonality property of sine functions, we obtain

\[ C_u = \frac{2}{(\pi)^{\frac{1}{2}}} \sin \frac{\pi u}{2} \left( \begin{array}{c} \frac{H_{p + 2, q + 2}^{m, n + 2}}{H_{p, q}^{m, n + 2}} \left[ z \left| \begin{array}{c} \left( \frac{1 - \rho}{2}, \delta \right), \left( -\rho/2, \delta \right), (a_p, e_p) \\ (b_q, f_q), \left( -\frac{\rho + u}{2}, \delta \right), \left( -\frac{\rho - u}{2}, \delta \right) \end{array} \right| \right] \end{array} \right) \]

(12)

From (11) and (12), the formula (8) follows immediately.

**PARTICULAR CASES**

In (7), assuming \( \delta \) as a positive integer, putting \( e_j = f_i = 1 \) (\( j = 1, \ldots, p \); \( i = 1, \ldots, q \)), using the formula

\[ H_{p, q}^{m, n} \left[ z \left| \begin{array}{c} (a_p, 1) \\ (b_q, 1) \end{array} \right| \right] = G_{p, q}^{m, n} \left[ z \left| a_p \right| b_i \right], \]

and simplifying with the help of (1), (4) and (9), we get a result recently obtained by Bajpai, viz.

\[ \left( \sin \theta \right)^p G_{p, q}^{m, n} \left[ z (\sin \theta)^{2\delta} \left| \begin{array}{c} a_p \\ b_q \end{array} \right| \right] = \frac{1}{(\pi \delta)^\frac{1}{2}} G_{p + \delta, q + \delta}^{m, n + \delta} \left[ z \left| \Delta \left( \frac{1 - \rho}{2} \right), a_p \right| b_q, \Delta \left( \delta, -\rho/2 \right) \right] \]

\[ + \frac{2}{(\pi \delta)^\frac{1}{2}} \sum_{r=1}^{\delta} G_{p + 2\delta, q + 2\delta}^{m, n + 2} \left[ z \left| \Delta \left( 2\delta, -\rho \right), \Delta \left( \delta, -\rho/2 \right) \right| b_q, \Delta \left( \delta, -\rho + r \right), \Delta \left( \delta, -\frac{\rho - r}{2} \right) \right] \]

\[ \times \cos \frac{\pi r}{2} \cos r\theta, \]

(13)

where \( 2(m + n) > p + q \), \( | \arg z | < (m + n - \frac{1}{2} p - \frac{1}{2} q) \pi \),

\[ \text{Re} (2 \delta b_j) > -\rho - 1 (j = 1, \ldots, m), 0 < \theta < \pi. \]

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**REFERENCES**