Librations of a Quasi-Lagrangian Gyro

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ABSTRACT

The proximate motion of a Lock-Fowler projectile with \( q > 0 \) is studied via geometrical method due to Copeland. Steady and unsteady motions are classified taking into account conditions at launch.

1. INTRODUCTION

The librations of a quasi-Lagrangian gyro as propounded by Lock and Fowler\(^1,2\) have been geometrically analyzed by Rath and Namboodiri\(^3\). The qualitative analysis of the librational motions were then analyzed by using the classical method due to Routh\(^4\). The results obtained did not have a direct reference to the initial conditions of motion. A graphical representation as adopted by Copeland\(^5\) for analysing the motion of a Lagrangian gyro is now adopted here for studying the entire history of motions of the quasi-Lagrangian gyro\(^*\). With the aid of Osgood’s\(^6\) intrinsic equations extended to the present case, the space curve (signifying the total motion) that could be considered as a trace on a unit sphere is discussed in detail with a set of typical examples. The dependance of the nature of motion on the initial conditions has been shown and how it bifurcates from one type of motion to another when they (initial conditions) are continuously varied has been exemplified. It is also shown in the last section that the Lagrangian gyro is a special case of the present system.

2. EQUATIONS OF MOTION

In analogy with Osgood’s equations for a Lagrangian gyroscope one can write the intrinsic equations of motion of a quasi-Lagrangian gyro (in Copeland’s form) as follows:

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* A Lock-Fowler projectile (with \( q > 0 \)) is here defined to be quasi-Lagrangian gyro.
Here $A$ and $B$ are respectively the axial and transverse moments of inertia of the gyro and $N$ denotes the angular spin about the symmetrical axis. $\delta$ and $\phi$ are the colatitude and longitude of a point $P$ on the axis, where it intersects a unit sphere. If $\Gamma$ be the trace of $P$ on the unit sphere then $k$ denotes its geodesic curvature at $P$ and $v$ is the instantaneous velocity of $P$. A dot (.) is used to represent differentiation with respect to time whereas a prime (') is used to denote differentiation with respect to the arc length $a$ of $\Gamma$. The parameters $q$ and $s$ appearing in the expression for $\mu(\delta)$ are certain constants.  

From Eqn. (3), we have

$$AN = B\Omega$$

and on integration Eqn. (1) yields

$$v^2 + \Omega^2 (1 - 4qs) \cos \delta/2s + q\Omega^2 \cos^2 \delta = E$$

where $E$ is the energy integral given by

$$E = V_0^2 + \Omega^2 (1 - 4qs) \cos \delta_0/2s + q\Omega^2 \cos^2 \delta_0$$

with $\delta_0$ as the initial value of $\delta$ when time $t = t_0$.

From Eqns. (5) and (6), we get

$$v^2 = v_0^2 + \Omega^2 (1 - 4qs)(\cos \delta_0 - \cos \delta)/2s + q\Omega^2(\cos^2 \delta_0 - \cos^2 \delta)$$

Since $P$ moves on the unit sphere, we have

$$v^2 = \dot{x}^2 + \dot{\phi}^2 \sin^2 \delta$$

when $t = t_0$ Eqn. (8) assumes the value

$$v_0^2 = \dot{x}_0^2 + \dot{\phi}_0^2 \sin^2 \delta_0$$

The angular momentum integral gives

$$\dot{\phi} \sin^2 \delta + \Omega \cos \delta = F$$

If $\dot{\phi} = \dot{\phi}_0$ when $t = t_0$, $F$ can be expressed as

$$F = \dot{\phi}_0 \sin^2 \delta_0 + \Omega \cos \delta_0$$

Therefore, setting

$$x = \cos \delta$$

$$a = \Omega^2 (1 - 4qs)/2s$$

$$\beta = q\Omega^2$$

$$r = F/\Omega$$
and using Eqns. (6) to (11), we have
\[ x^2 = -\Omega^2 (r - x)^2 + (E - \alpha x - \beta x^2) (1 - x^2) = f(x) \]
and
\[ \dot{\phi} (1 - x^2) = \Omega (r - x) \]
Using Eqns. (12) to (15) in Eqns. (6) and (11) \( E \) and \( r \) can be rewritten as
\[ E = \phi_0^2 + \phi_0^2 (1 - x_0^2) + \alpha x_0 + \beta x_0^2 \]
\[ r = \phi_0 (1 - x_0^2) / \Omega + x_0 \]
where \( x_0 = \cos \delta_0 \) when \( t = t_0 \)

It can be seen that the roots \( x_i \) \((i = 1, 2, 3, 4)\) of \( f(x) = 0 \) follow the following distribution:
\[ x_1 \leq -1 \leq x_2 \leq x_3 \leq +1 \leq x_4 \] (20)

Evidently, the real motion of \( P \) is bounded between the two levels of limiting motion corresponding to the two real roots \( x_2 \) and \( x_3 \) (= \( x_0 \), say) that lie in the interval \([-1, +1]\).

3. GRAPHICAL REPRESENTATION OF MOTION

The entire motion is characterised by Eqns. (16) and (17). The parameters that control the motion are \( x_0, \phi_0, q, s \) and \( \Omega \). For a given value of \( q, s, \Omega \) and \( x_0 \) it can be seen that the nature of motion changes continuously from one type to another by varying \( \phi_0 \). Thus a change in \( \phi_0 \) would imply a change in a particular type of motion, say, from a wavy structure of \( \Gamma \) to a direct-retrograde type of motion represented by loops. This situation (and any other such situations) can be identified by a certain sets of curves in the \( \phi_0-x \) plane.

In the \( \phi_0-x \) plane, \( x = x_i \) \((i = 0, 1, 2, 4)\) are plane curves of which \( x = x_0 \), in particular, is a straight line. However, owing to the relation (20), we are interested only in the curve \( x = x_2 \), which is bounded by \( x = \pm 1 \). The region lying between the curve \( x = x_2 \) and \( x = x_0 \) thus, represents the real motion and for a given value of \( \phi_0 \), \( \Gamma \) takes specific shape, which lies entirely within the parallels of latitude given by \( x = x_0 \) and \( x = x_2 \). As \( x = x_2 \) and \( x = x_0 \) correspond to situations where \( \dot{\delta} \) vanishes, similar situations like vanishing of \( \phi_0 \), vanishing of geodisic curvature \( k \) and vanishing of the derivative of \( k \) (with respect of arclength \( \sigma \) of \( \Gamma \)) are respectively represented by the straight line \( x = x_p \), the curve \( x = x_g \) and the curve \( x = x_k \) (these curves are simple consequences of Eqns. (16), (19) and (2) for \( \dot{x}^2 = 0, r = 0, k = 0 \) and \( k' = 0 \). (cf. Appendix for derivations). These curves are given by

\[ x = x_i \equiv f(x) / (x - x_0) = \beta x^3 + (\alpha + \beta x_0) x^2 - (\phi_0^2 h + \beta + \Omega^2) x \\
- \phi_0^2 h x_0 + 2 \Omega h \phi_0 + \Omega^2 x_0 - \alpha - \beta x_0 \]

* This being an autonomous system, without loss of generality, we can assume that \( f(x_0) = 0 \), which implies only a proper choice of the initial conditions. Hence, here after, we presume that \( f(x_0) = 0 \).
where

\[ h = -x_0^2. \]

The entering of these curves in the region bounded by \( x = x_0 \) and \( x = x_\alpha \) indicates a qualitative change in the nature of motion. Analysing these curves graphically the nature of \( \Gamma \) can be studied with the help of corresponding rosette diagrams.

Two such typical sets of curves in \( \phi - x \) plane are shown in Figs. 1 - 3.

4. Classifications of Motion

The entire set of motion of a quasi-Lagrangian gyro can now be classified with the help of various curves plotted in the \( \phi - x \) plane for any given set of initial conditions. In the following paragraphs, we explain, the dependance of motion on the initial conditions, with the help of Figs. 1 and 3. In each case it is discussed, the corresponding rosette diagram (as a specimen) is also given.

4.1 Direct Motion

A region free from \( x = x_p \) always corresponds to direct motion as \( \dot{\phi} \) does not change its sign at any instant of time. In addition, if such a region contains the curve

![Figure 1. Classification of motion (\( \Omega = 1.0; X_\alpha = 0.35; \delta_\alpha = 69.56^\circ; a = 0.18; q = 0.05; \beta = 0.05 \) and \( s = 1.79 \)).](image-url)
Librations of a Quasi-Lagrangian Gyro

Figure 2. Classification of motion: enlarged view at X of Fig. 1.

\( x = x_k \) this would imply an inflection point for \( \Gamma \) and under such situations, the curve \( \Gamma \) will be wavy. On the other hand, if the region is free from both \( x = x_p \) and \( x = x_q \), it would naturally imply direct motion with non-wavy structure for \( \Gamma \). If \( x = x_k \) enters the region then \( k \) will be non-monotonous. Thus in the Figs. 1 and 2 direct and wavy motion is represented by regions between \( P_3 \) to \( P_5 \) and \( P_7 \) to \( P_{10} \) and similarly regions between \( P_1 \) to \( P_2 \), \( P_5 \) to \( P_7 \) \( P_{11} \) to \( P_{13} \) represents direct and non-wavy motion, since \( x = x_k \) does not enter these regions. As it is clear from the figures that \( k \) will be non-monotonous only in the regions \( P_3 \) to \( P_4 \), \( P_6 \) to \( P_9 \) and \( P_{13} \) and \( P_{14} \). (See Figs. 4 and 5 for rosettes).

Figure 3. A case of cusp at the level of maximum yaw \((\Omega = 1.0; X_\phi = 0.93; \delta_\phi = 22^\circ; a = 8.99; q = \beta = 4.65\) and \( s = 1.67 \)).
4.2 Direct and Retrograde Motion (Loops)

A change in the sign of longitudinal velocity of $P$ during its motion from one limiting level to another gives a direct and retrograde motion tracing continuously loops on the unit sphere. Necessarily $\dot{\phi}$ vanishes for a certain value of $x$ in $(x_2, x_0)$. Such a situation is characterised in the $\phi_0-x$ plane, whenever $x = x_p$ enters the region of real motion. A typical situation is shown in Figs. 1 and 2 for the regions between $P$, to $P_1$, and $P_{III}$ to $P_{II}$. Rosette diagram is given in Fig. 6.
4.3 Cuspidal Motion

If at certain points, \( \Gamma \) has got cusps then at such points both the longitudinal and latitudinal velocities must vanish. In the \( \hat{\phi}_0-x \) plane, this situation arises when the curves \( x = x_2 \) or the straight line \( x = x_0 \) intersect \( x = x_p \). Points like \( P_3 \) and \( P_{10} \) in Fig. 1 and 2 refer to such situations. The point \( P_3 \) for instance corresponds to initial point of motion. At this point, both the velocities (longitudinal and latitudinal) are zero, as \( x = x_p \) intersects \( x = x_0 \) and hence \( \Gamma \) is a curve with cusps. The first cusp occurs at this initial level and others occurring periodically thereafter on the same level. Although the curve \( x = x_{e} \) passes through this point it does not admit the point of inflection as it corresponds to cuspidal motion. Corresponding rosette is shown in Fig. 7.

Figure 6. Direct and retrograde motion (loops).

Figure 7. Cuspidal motion.
It may be noted that in all these three cases, the cusps occur at the level of minimum yaw. However, in the case of quasi-Lagrangian gyro, cusps can also occur at the level of maximum yaw. This situation is depicted by the point Y in Fig. 3 and corresponding rosette is shown in Fig. 8.

Figure 8. Cusp at the level of maximum yaw.

4.4 Motion Through the Poles

This type of motion usually represented by psuedo-loops occur at \( x_2 = \pm 1 \) where \( x = 0 \). However, \( x = 0 \) is not the consequence of \( \dot{\delta} = 0 \) but is due to \( \delta = 0 \) (or \( \pi \)). At \( x_2 = \pm 1 \), the curve passes through the poles continuously. These situations in \( \dot{\phi}_0-x \) plane are shown by points \( P_2 \) and \( P_{11} \) in Fig. 1.

4.5 Steady Precession

The points in the graph Fig. 1 corresponding to steady precessional motion in \( \dot{\phi}_0-x \) plane are the points where \( x = x_0 \) and \( x = x_2 \) intersect. The straight line \( x = x_0 \) intersects the curve \( x = x_2 \) in \( P_6 \) and \( P_{12} \) in Figs. 1 and 2. At these points the parallels of latitude \( x = x_0 \) and \( x = x_2 \) coincide to form a single parallel of latitude and reduces to this circle itself thus, exhibiting steady conical motion of the gyroscope.

4.6 Spiral Motion

Motion of this type takes place when the curve \( x = x_4 \) intersects the curve \( x = x_2 \) at \( x = +1 \). The roots \( x = x_4 \) and \( x = x_2 \) both become unity. It takes infinite time for \( P \) to reach at \( x_2 = x_4 = 1 \). This situation is depicted in Fig. 9 at \( \dot{\phi}_0 = 0.67 \). The corresponding rosette will be as in Fig. 10.

A summary of classification of motion is given in Table 1.

5. SPINNING TOP: (THE LAGRANGIAN GRAVITY GYRO)

It is easily seen that the equations of motion derived in Eqns. (16) and (17) become those of a spinning top when we set \( q = 0 \). Thus for \( q = 0 \) Eqns (21) to (24) become
Figure 9. A case of spiral motion ($\Omega = 1.0; X_0 = 0.50; \delta_0 = 60^\circ; \alpha = 0.27; q = 0.27; \beta = 0.27$ and $s = 0.63$).

Figure 10. Spiral motion.

\[
x = x_1 = a x^2 + (-\dot{\phi}_0^2 h - \Omega^2)x + (-\dot{\phi}_0^2 h x_0 + 2\dot{\phi}_0 h \Omega + \Omega^2 x_0 - \alpha)
\]

\[
x = x_p = \dot{\phi}_0 h / \Omega + x_0
\]

\[
x = x_q = (2 h \Omega \dot{\phi}_0^2 - a h \dot{\phi}_0 + a \Omega x_0) / a \Omega
\]

\[
x = x_k = 2 h \dot{\phi}_0^2 - (3 a h / 2 \Omega) \dot{\phi}_0 + a(x_0 - x) / 2
\]

where

\[ h = (1 - x_0^2) \]
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Table 1. Summary of classification of motion

<table>
<thead>
<tr>
<th>Sr. No.</th>
<th>Region/points in Figs. 1,2,3,9</th>
<th>Nature of motion</th>
<th>Shade used</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>P₁ - P₂</td>
<td>Direct – nonwavy</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>Direct – wavy</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>P₃ - P₄</td>
<td>Direct – wavy</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>P₃ - P₄</td>
<td>Direct – wavy</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>P₅ - P₆</td>
<td>Direct – wavy</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>P₆ - P₇</td>
<td>Direct – wavy</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>P₇ - P₈</td>
<td>Direct – wavy</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>P₈ - P₉</td>
<td>Direct – wavy</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>P₉ - P₁₀</td>
<td>Direct – wavy</td>
<td></td>
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<tr>
<td>10</td>
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<td></td>
</tr>
<tr>
<td>11</td>
<td>P₁₁ - P₁₂</td>
<td>Direct – wavy</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>P₁₂ - P₁₃</td>
<td>Direct – wavy</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>P₁₃ - P₁₄</td>
<td>Direct – wavy</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>P₁₄ - P₁₅</td>
<td>Direct – wavy</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>P₃, P₁₀</td>
<td>Cusps at the level of min. yaw</td>
<td>Nil</td>
</tr>
<tr>
<td>16</td>
<td>P₆, P₁₂</td>
<td>Steady precession</td>
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</tr>
<tr>
<td>17</td>
<td>P₂, P₁₁</td>
<td>Psuedo loops</td>
<td>Nil</td>
</tr>
<tr>
<td>18</td>
<td>Y</td>
<td>Cusps at the level of max. yaw</td>
<td>Nil</td>
</tr>
<tr>
<td>19</td>
<td>Z</td>
<td>Spiral (asymptotic) motion</td>
<td>Nil</td>
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</table>

Evidently, these are identical with Eqns. (8), (17), (18) and (21) of Copeland. A typical set of curves when \( q = 0 \) is given in Fig. 11 for ready reference.

6. CONCLUSION

By adopting Copeland's analysis, we have succeeded in classifying the various types of angular motion of a Lock-Fowler projectile. The classification depends on initial conditions chosen contrary to what has been done by Rath and Namboodiri earlier.
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REFERENCES

APPENDIX

(1) The Curve: \( x = x_k \)

The expression for \( k \) can be obtained by solving Eqn. (2) for \( k \); that is

\[
k = \left\{ ANv - \mu(\delta) \sin^2 \phi' \right\} / Bv^2
\]

It follows by substitution, that

\[
k = \Omega \left\{ v^2 - \Omega^2 \left( 1 - 4qs + 4qs \cos \delta \right) \sin^2 \phi' / 4s \right\} / v^3
\]

Using relations stated in Eqns. (12) to (19), one gets

\[
k = \Omega \left\{ \left( 1 - x_0^2 \right) \phi_0^2 - (a+2\beta x) \left( 1 - x_0^2 \right) \phi_0 / 2\Omega \right\} / v^3 \tag{A.3}
\]

If \( k = 0 \) and \( v \) is not infinite, Eqn. (A.3) becomes

\[
x = \frac{2\Omega \left[ \left( a+2\beta x \right) x_0 / 2 + \left( \phi_0 - a/2\Omega \right) h\phi_0 \right]}{\Omega \left( a+2\beta x \right) + 2\beta h \phi_0}
\]

which can be expressed as

\[
x = x_g \equiv x_0 + \frac{\Omega}{h} \left\{ \left[ \phi_0 - (a+2\beta x) / 2\Omega \right] \right\} \tag{A.4}
\]

(2) The Curve: \( x = x_k \)

An expression for \( x = x_k \) can also be derived as follows.

Using Eqn. (29), Eqn. (A.3) can be written as:

\[
k = \Omega \left\{ \phi_0^2 h - (a+2\beta x) h\phi_0 \phi_0 / 2\Omega + (a+2\beta x_0) \left( x_0 - x \right) / 2 \right\} / v^3 \tag{A.5}
\]

Differenting \( k \) with respect to arc length \( \sigma \), we get

\[
\frac{v^6}{\Omega} k' = v^3 \left\{ \left( -\beta h\phi_0 / \Omega \right) \frac{dx}{d\sigma} \left[ \left( a+2\beta x_0 / 2 \right) \frac{dx}{d\sigma} \right] \right\}
\]

\[
\left[ \phi_0^2 h - (a+2\beta x) h\phi_0 / 2\Omega + (a+2\beta x_0) \left( x_0 - x \right) \right] 3v^2 \frac{dv}{d\sigma} \tag{A.6}
\]

Using Eqns (13) and (14) in Eqn. (5), we have

\[
v^2 = E - ax - \beta x^2
\]

we also have

\[
\frac{dx}{d\sigma} = \frac{dx}{dt} \frac{dt}{d\sigma} = \frac{x}{v}
\]
\[ 3v^2 \frac{dv}{d\sigma} = 3v^2 \frac{dv}{dx} \frac{dx}{dt} \frac{1}{d\sigma/dt} \]

Using above relations (A.6) becomes:

\[
\frac{v^2}{\Omega} k' = - (E - ax - \beta x^2) \left[ (\beta h/\Omega) \dot{\phi}_0 + (a+2\beta x_0)/2 \right] \dot{x} + 3 \left\{ h \dot{\phi}_0^2 - (a+2\beta x) (h/2\Omega) \phi_0 + (a+2\beta x_0) (x_0 - x)/2 \right\} \frac{(a+2\beta x) \dot{x}}{2}
\]

Therefore

\[
k' = \Omega \dot{x} \left\{ 3 (a+2\beta x) \left[ \phi_0^3 h - (a+2\beta x) (h/2\Omega) \phi_0 + (a+2\beta x_0) (x_0 - x)/2 \right] - (E - ax - \beta x^2) \left[ (2\beta h/\Omega) \phi_0 + (a+2\beta x_0) \right] \right\}
\]

\[
E = \dot{\phi}_0^2 h + ax_0 + \beta x_0^2
\]

We have

\[ k' \equiv \Omega \dot{x} \psi (\dot{\phi}_0, x) \]

where

\[
\psi (\phi_0, x) = (-2\beta h^2/\Omega) \dot{\phi}_0^3 + 2h (3\beta x - \beta x_0 + a) \phi_0^2 + (4\beta^2 x^2 + 4a\beta x + 2\beta^2 x_0^2 + 2a\beta x_0 + 3a^2/2) \frac{h}{\Omega} \dot{\phi}_0 + \frac{3}{2} (a+2\beta x_0) (a+2\beta x) (x_0 - x) - (a+2\beta x_0) \left[ a (x_0 - x) + \beta (x_0^2 - x^2) \right]
\]

\[ k' \] vanishes, when \( \psi (\phi_0, x) = 0 \). Therefore, we have,

\[
x = x_k \equiv (-2\beta h^2/\Omega) \dot{\phi}_0^3 + 2h (3\beta x - \beta x_0 + a) \phi_0^2 + (4\beta^2 x^2 + 4a\beta x + 2\beta^2 x_0^2 + 2a\beta x_0 + 3a^2/2) \frac{h}{\Omega} \dot{\phi}_0 + (a+2\beta x_0) (x_0 - x) (2\beta x - \beta x_0 + a/2) \tag{A.7}
\]

which signifies vanishing of \( k' \)